Critical exponents, the spectrum of group extended transfer operators and Kazhdan distance

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Declarations

This thesis is the culmination of the author's period of registration as a PhD student at the University of Warwick. The results of Chapter 2 were in collaboration with Richard Sharp, and have been published [14]. The results of Chapter 3 are due to the author and appear in a preprint [13]. Similarly, Chapter 4 is due to the author, and will be produce as a preprint in the future. It should also be noted that some of the background exposition is based on some excellent monographs and papers, and is indicated where appropriate.

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Abstract

For a group of isometries Γ of a simply connected 'negatively curved' space X, the critical exponent δ_Γ is the exponential growth rate of the Γ-orbit of a point. We consider separately the cases where X is a pinched Hadamard manifold and where X is CAT(-1). The properties of the action of a cocompact, or convex cocompact group Γ_0 are relatively well-understood. We consider the case where Γ is a normal subgroup of such a Γ_0.

In the case that X is a pinched Hadamard manifold, we show that $\delta_\Gamma = \delta_{\Gamma_0}$ if and only if Γ_0/Γ is amenable. In addition, for any family of normal subgroups $\mathcal{N}$ of Γ_0, we show that the gap $\delta_\Gamma < \delta_{\Gamma_0}$ is uniform in Γ if and only if the Kazhdan distances of the permutation representation of Γ_0/Γ are bounded away from zero. These are analogues of results of Brooks [6, 7] and Sunada [52] on the spectrum of the Laplacian.

The proofs rely on the spectral properties of transfer operators given by the symbolic dynamics for the geodesic flow, as formulated by Bowen [5] and Ratner [41]. We give an analogous result for the spectral radius of group extended transfer operators in terms of Kazhdan distances. This builds on recent work of Stadlbauer [50] and Jaersich [22].

In the case that X is CAT(-1), we use symbolic dynamics given by the Markov grammar (see [17]) for Γ_0, as formulated by Bourdon [3], Lalley [27], and Pollicott and Sharp [39, 40]. For a class of Γ_0 with a more tractable structure, we show that $\delta_\Gamma < \delta_{\Gamma_0}$ when Γ_0/Γ is non-amanable; and moreover there is a gap uniform in Γ if and only if the Kazhdan distance of the permutation representations of Γ_0/Γ are bounded away from zero.

Chapter 1

Introduction

This thesis is the product of the author's PhD registration at the University of Warwick. The mathematical exposition of each chapter is self-contained, and the ordering of the chapters reflects the chronology of the research. The results of Chapter 2 are in collaboration with Richard Sharp, and have been published in Mathematische Annalen [14]. The results of the remaining chapters are due to the author. The results of Chapter 3 are contained in a pre-print [13].

1.1 Dynamics, geometry, groups and growth

A classically studied dynamical system is the geodesic flow $\phi_0^t : SM_0 \to SM_0$ on the unit tangent bundle of a compact n-dimensional hyperbolic manifold M_0. There are countably many periodic orbits γ for the geodesic flow, and $1 < \limsup_{T\to\infty} \#\{\gamma : |\gamma| \le T\}^{1/T} < \infty$, where $|\gamma|$ indicates the (least) period of the orbit. Moreover, there is an asymptotic for counting prime (meaning traversed once) periodic orbits: $(n-1)Te^{-(n-1)T}\#\{\gamma : \gamma \text{ prime}, |\gamma| \le T\} \to 1$ as $T \to \infty$. We will be interested in the more general case where M_0 is a manifold with pinched strictly negative sectional curvatures and $\phi_0^t : SM_0 \to SM_0$ has a non-empty compact non-wandering set. In this case, there is an asymptotic $h_0Te^{-h_0T}\#\{\gamma : \gamma \text{ prime}, |\gamma| \le T\} \to 1$ as $T \to \infty$, where $h_0 = h(M_0)$ is the topological entropy of the flow [33, 36] (see section 1.2 for the definition of topological entropy). The result can be proven with *symbolic dynamics* – this is an important area that we expand on in section 1.2.

The focus of this thesis will is a particular family of geodesic flows (always arising from a manifold M with pinched, strictly negative sectional curvatures) for which $\phi^t : SM \to SM$ has a non-compact non-wandering set. Namely, the geodesic flow arising from infinite regular covers M of M_0 (with M_0 as above). Equivalently, M is a manifold with a discrete group of isometries G such that $M/G = M_0$; and when G is infinite the non-wandering set is non-compact. The growth associated to M is given by the growth of ϕ-periodic orbits γ that intersect a non-empty, open, relatively compact $W \subset SM$. Define

$$h(M) = \limsup_{T\to\infty} \frac{1}{T} \log \#\{\gamma : \gamma \cap W \neq \emptyset, |\gamma| \leq T\}.$$

(The definition is independent of W.)

We now introduce another important object of this thesis: amenable groups. A (countable) group G is *amenable* if $\ell^\infty(G)$ admits an invariant mean, i.e. that there exists a bounded linear functional $\nu : \ell^\infty(G) \to \mathbb{R}$ such that, for all $f \in \ell^\infty(G)$,

1. $\inf_{g\in G} f(g) \leq \nu(f) \leq \sup_{g\in G} f(g)$; and
2. for all $g \in G$, $\nu(f_g) = \nu(f)$, where $f_g(h) = f(g^{-1}h)$.

The concept was introduced by von Neumann in 1929. One sees immediately from this definition that finite groups are amenable by taking

$$\nu(f) = \frac{1}{|G|} \sum_{g\in G} f(g).$$

There is already a well established relationship between amenability and spectral geometry given by Brooks [6, 7]. Let $M \to M_0$ be a regular covering of a Riemannian manifold M_0 of "finite topological type" (i.e. M_0 is the union of finitely many simplices). Let $\lambda_0(M)$ and $\lambda_0(M_0)$ denote the bottom of the spectrum of the Laplacian on M and M_0 respectively. Brooks shows that $\lambda_0(M) = \lambda_0(M_0)$ if and only if the group of deck transformation given by the covering is amenable. We will refer to a result of this form as *an amenability dichotomy.*

Chapter 2 gives an amenability dichotomy for $h(M) = h(M_0)$.

Theorem 1.1.1 (Dougall-Sharp [14])**.** *We have $h(M) = h(M_0)$ if and only if G is amenable.*

Note that the implication G amenable $\implies h(M) = h(M_0)$ is due to Roblin [43]. A full history of this result is given in Chapter 2.

In order to state the remainder of our results, we will find it useful to reformulate the result in the preceeding paragraphs with an equivalent viewpoint. Let X be a connected, simply connected and complete Riemannian manifold with sectional curvatures bounded between two negative constants. We call such an X a *pinched Hadamard manifold.* Let Γ be a *non-elementary,* torsion-free and *convex co-compact* group of isometries of X (see Chapter 2 for the definition) and let $\Gamma \trianglelefteq \Gamma_0$ be a normal subgroup. We define the *critical exponent* of Γ to be the abscissa of convergence of the Poincaré series

$$\eta_\Gamma(s) = \sum_{g\in\Gamma} e^{-sd_X(x,gx)},$$

for any choice of base point $x \in X$, and denote it by δ_Γ. Writing $M_0 = X/\Gamma_0$ and $M = X/\Gamma$, we have that $\delta_{\Gamma_0} = h(M_0)$ and $\delta_\Gamma = h(M)$. Therefore $\delta_\Gamma < \delta_{\Gamma_0}$ when Γ_0/Γ is non-amenable. The results of Chapter 3 describe how this gap behaves for an arbitrary family of normal subgroups.

Our result is to describe coarse behaviour of δ_Γ, over any family $\mathcal{N}$ of normal subgroups of Γ_0, in terms of *Kazhdan distances* associated to the quotients Γ_0/Γ, which we explain below. Some natural families of coverings are a tower of regular covers $M_1 \to M_2 \to \cdots \to M_0$, corresponding to a family $\Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_0$ of normal subgroups of Γ_0; and the family of all non-amenable regular covers of M_0, i.e. all $\Gamma \trianglelefteq \Gamma_0$ for which Γ_0/Γ is non-amenable.

In the following, G is assumed to be a countable group (however, many of the definitions can be made in the setting of locally compact groups). We give an equivalent formulation of amenability due to Følner [16]. A group G is *amenable* if and only if for every $\epsilon > 0$, and for every finite set A, there exists a set E which is $\epsilon, A-$invariant; that is,

$$\#E\Delta Ea \leq \epsilon\#E,$$

for all $a \in A$.

Write $\mathbb{1}_E \in \ell^2(G)$ for the indicator function on the set E. Noting that $\#E\Delta Ea = |\mathbb{1}_E - \mathbb{1}_{Ea}|$, there is the following equivalent definition in terms of the right regular representation $\pi_G : G \to \mathcal{U}(\ell^2(G))$, $(\pi_G(g)f)(x) = f(xg)$, due to Hulanicki [21]. A group G is *amenable* if and only if, for any finite generating set $A \subset G$,

$$\inf_{v\in\ell^2(G),|v|=1} \max_{a\in A} |\pi_G(a)v - v| = 0.$$

For any unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ in a Hilbert space $(\mathcal{H}, |\cdot|)$, the quantity $\kappa_A(\rho, \mathbb{1})$ defined by

$$\kappa_A(\rho, \mathbb{1}) := \inf_{v\in V,|v|=1} \max_{a\in A} |\rho(a)v - v|$$

is called the *Kazhdan distance (between ρ and the trivial representation $\mathbb{1}$)*. A group is said to have *Property* (T) if there is some $\kappa > 0$ such that $\kappa_A(\rho, \mathbb{1}) > \kappa$

for all unitary representations that have no invariant vector.

There is an analogue of Brooks' result for the Laplacian in this setting. We assume now that M_0 is compact, and so $\lambda_0(M_0) = 0$, and write $M_0 = X/\Gamma_0$. For any $\Gamma \trianglelefteq \Gamma_0$ we get a regular cover $M_\Gamma = X/\Gamma$ of M_0. Sunada shows that for any finite generating set A of Γ_0, there are constants c_1, c_2 depending only on the geometry of X and on A, such that for any regular cover $M_\Gamma = X/\Gamma$, we have

$$c_1(\kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}))^2 \leq \lambda_0(M) \leq c_2(\kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}))^2,$$

where A/Γ denotes the projection of A to the quotient Γ_0/Γ. These results were also generalised by Roblin and Tapie [44], and in the thesis of Tapie [53], relating the difference $\lambda_0(M_0) - \lambda_0(M)$ to the bottom of the spectrum of a combinatorial Laplacian (which is in turn related to the Kazhdan distance).

A consequence of these spectral results is that, for any family of normal subgroups $\mathcal{N}$ of Γ_0, we have

$$\inf_{\Gamma \in \mathcal{N}} \lambda_0(X/\Gamma) = 0 \text{ if and only if } \inf_{\Gamma \in \mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) = 0.$$

For n-dimensional real hyperbolic space $\boldsymbol{H}^n_{\mathbb{R}}$, the spectral geometry and dynamics are related by a celebrated theorem due to Patterson [34] and Sullivan [51]. We have that,

$$\lambda_0(X/\Gamma) = \begin{cases} \delta_\Gamma(n-1-\delta_\Gamma) & \text{if } \delta_\Gamma \geq \frac{n-1}{2} \\ \frac{(n-1)^2}{4} & \text{if } \delta_\Gamma \leq \frac{n-1}{2}. \end{cases}$$

There are analogous statements for the other noncompact rank 1 symmetric spaces. However these methods fail to extend to spaces which do not satisfy such strong symmetry hypotheses.

It is known that the isometry group of $\boldsymbol{H}^n_{\mathbb{R}}$ does not have Property (T), and so its cocompact subgroups also do not satisfy Property (T). However, when we consider groups arising from variable curvature, we do find cocompact examples with Property (T). For n-dimensional quaternionic hyperbolic space $\boldsymbol{H}^n_{\mathbb{H}}$, with $n \geq 2$, and the Cayley plane $\boldsymbol{H}^2_{\mathbb{O}}$, Corlette [11] obtains a more precise description of the critical exponents, using the fact that the isometry group of $\boldsymbol{H}^n_{\mathbb{H}}$ and $\boldsymbol{H}^2_{\mathbb{O}}$ have Property (T). If Γ is a lattice in $\mathrm{Isom}^+(\boldsymbol{H}^n_{\mathbb{H}})$, then $\delta_\Gamma = 4n+2$; and otherwise $\delta_\Gamma \leq 4n$. There is also the corresponding statement for the Cayley plane: $\delta_\Gamma = 22$ when Γ is a lattice in $\mathrm{Isom}^+(\boldsymbol{H}^2_{\mathbb{O}})$; and otherwise $\delta_\Gamma \leq 16$. Therefore, in each case, for a fixed lattice Γ_0, there is a uniform gap $\delta_\Gamma < \delta_{\Gamma_0}$ for infinite index $\Gamma \leq \Gamma_0$. The mechanism behind the gap in the critical exponents for $\boldsymbol{H}^n_{\mathbb{H}}$ and $\boldsymbol{H}^2_{\mathbb{O}}$, as shown by Corlette, is the fact their isometry groups have Property (T) [2]. The symmetry of the spaces $\boldsymbol{H}^n_{\mathbb{H}}$ and $\boldsymbol{H}^2_{\mathbb{O}}$ are notable in the approach to this problem.

The main theorem of Chapter 3 is the following.

Theorem 1.1.2. *Let Γ_0 be a convex cocompact group of isometries of a pinched Hadamard manifold X, and let A be a finite generating set for Γ_0. For any collection $\mathcal{N}$ of normal subgroups of Γ_0, we have*

$$\sup_{\Gamma \in \mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0} \text{ if and only if } \inf_{\Gamma \in \mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0.$$

If Γ_0 has Property (T), then we have that

$$\inf_{\Gamma \trianglelefteq \Gamma_0 : [\Gamma_0:\Gamma]=\infty} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0.$$

We remark that, in this case, $[\Gamma_0 : \Gamma] = \infty$ is equivalent to Γ_0/Γ being non-amenable.

Corollary 1.1.1. *With the hypotheses of Theorem 1.1.2, if Γ_0 has Property (T) then*

$$\sup \{\delta_\Gamma : \Gamma \trianglelefteq \Gamma_0, [\Gamma_0 : \Gamma] = \infty\} < \delta_{\Gamma_0}.$$

We conclude this discussion with geometries that do not possess such a smooth structure. Let X be a CAT(-1) space and let Γ_0 be a non-elementary cocompact group of isometries acting freely and properly discontinuously on X. An important case of a CAT(-1) space is the Cayley graph of the free group on $d \geq 2$ generators (with respect to its symmetric free basis S), and the metric is given by $d(g, h) = |g^{-1}h|_S$, the minimal length of the element g expressed as a word in S. In this case, $\delta_{F_d} = 2d - 1$, and for any $H \trianglelefteq F_d$,

$$\delta_H = \limsup_{N\to\infty} \frac{1}{N} \log \#\{g \in H : |g|_S \leq N\},$$

is the *relative growth* of H in F_d. Then a theorem of Grigorchuk [18] states that $\delta_H = \delta_{F_d}$ if and only if F_d/H is amenable. Moreover, it was conjectured [19] that the same result should hold for the Cayley graph of any hyperbolic group.

We have seen in Theorem 1.1.1, that when X is a Riemannian manifold, equality of critical exponents $\delta_\Gamma = \delta_{\Gamma_0}$ holds precisely when Γ_0/Γ is amenable. In the CAT(-1) setting, the result of Roblin [43] still applies: if Γ_0/Γ is amenable then $\delta_\Gamma = \delta_{\Gamma_0}$. This conclusion was also obtained by Sharp [48] in the case that Γ_0 is a free group.

Since Γ_0 acts co-compactly on a CAT(-1) space, it follows that Γ_0 is word hyperbolic. In this way we may associate to it a Markov grammar, which is roughly speaking, a (finite) directed graph, with a distinguished initial state, which writes the elements of Γ_0 uniquely (and with minimal length). This is a powerful tool which dates back to Cannon [9] in the case of cocompact discrete groups of isometries of real hyperbolic space. He uses this to show that the growth series $f_S(x) = \sum_{n=0}^{\infty} N(n)x^n$, $N(n) = \#\{g \in \Gamma_0 : |g|_S = n\}$, of such a group Γ_0 is a rational function.

Following Bourdon [3] and Lalley [27], this machinery was employed by Pollicott and Sharp [39] to study comparison theorems between the word length and geometric displacement of a point. And later, in the case of variable negative curvature, Pollicott and Sharp [40] showed that the Poincaré series of a discrete cocompact group of isometries Γ_0 has a meromorphic continuation to an $\epsilon > 0$ neighbourhood of the half-plane $\mathrm{Re}(s) > \delta_{\Gamma_0}$. Following [39], we say that the Markov grammar is *gregarious* if the directed graph has only one connected component that is not a singleton. These definitions will be made precise in Chapter 4. Our main result is the following.

Theorem 1.1.3. *Let Γ_0 be a cocompact group of isometries of a* CAT(-1) *space. Assume that Γ_0 admits a Markov grammar that is gregarious. For any collection $\mathcal{N}$ of normal subgroups of Γ_0, we have*

$$\inf_{\Gamma \in \mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0 \implies \sup_{\Gamma \in \mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0}.$$

In particular, for any normal subgroup $\Gamma \leq \Gamma_0$, we have that $\delta_\Gamma = \delta_{\Gamma_0}$ if and only if Γ_0/Γ is amenable

Remark 1.1.1. The class of groups with a Markov grammar that is gregarious includes the class of all cocompact Fuchsian groups [39], [47].

1.2 Symbolic dynamics, transfer operators and representations of groups

In this section we introduced subshifts of finite type and discuss dynamical properties of extensions by groups. The results relate the spectrum of group extended transfer operators to Kazhdan distances of associated group representations, and are instrumental in proving the theorems discussed in the previous section.

For a finite alphabet $\mathcal{W} = \{1, \ldots, k\}$, we can give rules governing when two letters in the alphabet can be concatenated in terms of a $k \times k$ matrix A with entries 0 or 1. Namely, for $i, j \in \mathcal{W}$, the concatenation ij is said to be *admissible* if $A(i, j) = 1$. In this way, the set of admissible words of length n,

$\mathcal{W}^n$, is the collection of concatenations $w = x_0 \cdots x_{n-1}$, where $x_0, \ldots, x_{n-1} \in \mathcal{W}$ and $A(x_i, x_{i+1}) = 1$ for all $i = 0, \cdots, n-2$. Extending this to one-sided infinite words, define the (one-sided) shift space to be

$$\Sigma^+ = \left\{ x^0 x^1 \cdots \in \mathcal{W}^{\mathbb{Z}^+} : \forall i \in \mathbb{Z}^+,\ A(x^i, x^{i+1}) = 1 \right\}.$$

The two-sided shift space Σ is defined analogously by

$$\Sigma = \left\{ \cdots x^{-1} x^0 x^1 \cdots \in \mathcal{W}^{\mathbb{Z}} : \forall i \in \mathbb{Z},\ A(x^i, x^{i+1}) = 1 \right\}.$$

For brevity, we make the following definitions for the two-sided space Σ. However, they pass to Σ^+ by the canonical projection $\Sigma \to \Sigma^+$, given by forgetting past (negative) coordinates. There is a natural dynamical system, $\sigma : \Sigma \to \Sigma$, called the *shift map*, that shifts the sequence left: $\sigma(x)^i = x^{i+1}$. Together, we call the pair (Σ, σ) a *subshift of finite type*. We always assume that σ is topologically mixing.

There is a natural metric on Σ under which the shift map σ is continuous. The functions $f : \Sigma \to \mathbb{R}$ that are Hölder continuous are seen to have good properties in relation to the dynamics.

For a strictly positive Hölder continuous function $r : \Sigma \to \mathbb{R}$, we form the suspension space Σ_r

$$\Sigma_r = \Sigma \times \mathbb{R} / \sim_r,$$

where $\sim_r$ is the equivalence relation $(x, s) \sim_r (\sigma x, s - r(x))$. We define the suspension flow $\sigma_r^t : \Sigma_r \to \Sigma_r$, locally by $\sigma_r^t(x, s) = (x, s + t)$. The *pressure* $P(f, \sigma)$ of a Hölder continuous $f : \Sigma \to \mathbb{R}$ is defined to be

$$P(f, \sigma) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\substack{x \in \Sigma: \\ \sigma^n x = x}} e^{f^n(x)}.$$

We remark that the form of this definition is special to our setting; a more detailed discussion of pressure, including the important variational principle, is given in [54]. There is a relation between the pressure and the *topological entropy* $h_{\mathrm{top}}(\sigma_r^t)$ of the suspension flow σ_r^t: namely $h_{\mathrm{top}}(\sigma_r^t) = s$ is the unique solution to $P(-sr, \sigma) = 0$. The *topological entropy* of a flow f^t on a compact metric space is defined to be,

$$\sup_{\delta > 0} \limsup_{T \to \infty} \frac{1}{T} \log \# M(T, \delta),$$

where $M(T, \delta)$ is the maximum cardinality of a T, δ-separated set.

Celebrated work of Bowen [5] and Ratner [41] allows us to relate the geodesic flow to a suspension flow (we make this precise in Chapters 2 and 3). In this way, we see that analysing the pressure of Hölder continuous functions will be crucial to proving the results described in section 1.1. In order to do this we introduce an operator which is of great importance in the field of dynamical systems.

We now specialise to the one-sided shift space Σ^+. (For any function $r : \Sigma \to \mathbb{R}$ there is a cohomologous function $r : \Sigma^+ \to \mathbb{R}$ which depends only on future coordinates. See Chapters 2 and 3.) For a Hölder continuous $r : \Sigma^+ \to \mathbb{R}$, we define the *transfer operator* $L_r : C(\Sigma^+, \mathbb{R}) \to C(\Sigma^+, \mathbb{R})$ by

$$L_r f(x) = \sum_{\substack{y \in \Sigma^+ : \\ \sigma y = x}} e^{r(y)} f(y),$$

where $C(\Sigma^+, \mathbb{R})$ is the Banach space of continuous functions with the supremum norm $\|\cdot\|_\infty$. We write $\mathrm{spr}(L_r)$ for the spectral radius of L_r in this Banach space, omitting explicit reference to the space. We find better spectral properties for L_r when we restrict to the smaller Banach space of Hölder continuous functions, in which case, by the Ruelle-Perron-Frobenius theorem [33], L_r has a simple, isolated, maximal eigenvalue at $e^{P(r,\sigma)}$.

For a countable group G, define the *group extension (with skewing function $\psi : \Sigma^+ \to G$)*, $T_\psi : \Sigma^+ \times G \to \Sigma^+ \times G$, to be the product space $\Sigma^+ \times G$ together with dynamical system

$$T_\psi(x, g) = (\sigma x, g\psi(x)^{-1}).$$

In the following ψ is always assumed to depend only on one letter. In this way, we can think of ψ as a function $\psi : \mathcal{W} \to G$. Moreover, for every $n \in \mathbb{N}$, we define $\psi^n : \Sigma^+ \to G$, $\psi^n(x) = \psi(x^0) \cdots \psi(x^{n-1})$, and write $\psi^n : \mathcal{W}^n \to G$, $\psi^n(w) = \psi(w^0) \cdots \psi(w^{n-1})$.

For $r : \Sigma^+ \to \mathbb{R}$, there is a unique $\widetilde{r} : \Sigma^+ \times G \to \mathbb{R}$ such that $\widetilde{r}(x, g) = r(x)$. We therefore dispense with the cumbersome tilde, and simply write this function as $r : \Sigma^+ \times G \to \mathbb{R}$. Define the *group extended transfer operator* $\mathcal{L}_r$ pointwise by

$$\mathcal{L}_r f(x, g) = \sum_{\substack{(y,g^*) \in \Sigma^+ \times G : \\ T(y,g^*) = (x,g)}} e^{r(y)} f(y, g^*).$$

Define the Banach space $(\mathcal{C}^\infty, \|\cdot\|)$ by

$$\mathcal{C}^\infty = \left\{ f \in C(\Sigma^+ \times G, \mathbb{R}) : \|f\| < \infty \right\},$$

$$\|f\| = \sqrt{\sum_{g \in G} \sup_{x \in \Sigma^+} |f(x,g)|^2}.$$

Then $\mathcal{L}_r : \mathcal{C}^\infty \to \mathcal{C}^\infty$ is a bounded operator. We will always take the spectral radius $\mathrm{spr}(\mathcal{L}_r)$ with respect to the $\mathcal{C}^\infty$ norm, and continue to omit reference to the space.

Following Sarig, [46], define the *Gurevič pressure* $P_{\mathrm{Gur}}(f, T_\psi)$ of a Hölder continuous $f : \Sigma^+ \times G \to \mathbb{R}$ by

$$P_{\mathrm{Gur}}(f, T_\psi) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x: \\ \psi^n(x) = e}} e^{f^n(x)},$$

where e is them identity of G.

Our general aim is to understand how the spectrum of the transfer operator $\mathcal{L}_r$ behaves as a function of the group.

A prototypical example of the relation between groups and spectra is the following theorem of Kesten from 1959, which inspired the earlier stated result of Brooks in spectral geometry. Let G be a countable group and let $p : G \to \mathbb{R}^+$ be a symmetric probability distribution (i.e. $\sum_{g \in G} p(g) = 1$ and $p(g^{-1}) = p(g)$ for all $g \in G$) such that its support, $\mathrm{supp}(p)$, generates G. This defines a symmetric random walk operator $M : \ell^2(G) \to \ell^2(G)$ by

$$Mf(x) = \sum_{g \in G} p(g) f(xg).$$

Let $\lambda(G, M)$ be the $\ell^2(G)$-spectral radius of M.

Theorem 1.2.1 (Kesten [25]). *We have $\lambda(G, M) = 1$ if and only if G is amenable.*

This was also extended by Ollivier to give a spectral characterisation of Property (T). We state a simplified version here. For $H \trianglelefteq G$ define $\rho_H : G \to \mathcal{U}(\ell^2(G/H))$ by $(\rho_H(g)f)(x) = f(xg)$. Let $p : G \to \mathbb{R}$ as above, and define random walk operators $M_H : \ell^2(G/H) \to \ell^2(G/H)$ for each $H \trianglelefteq G$ by

$$M_H f = \sum_{g \in G} p(g) \rho_H(g) f.$$

Let $\lambda(G, \rho_H)$ be the $\ell^2(G/H)$-spectral radius of M_H.

Theorem 1.2.2 (Ollivier [31]). *Let $\mathcal{N}$ be a collection of normal subgroups of G, and A be a generating set of G. We have $\sup_{H \in \mathcal{N}} \lambda(G, M) = 1$ if and only if $\inf_{H \in \mathcal{N}} \kappa_A(\rho_H, \mathbb{1}) = 0$.*

In special cases, the group extended transfer operator is isomorphic to a random walk operator: for example, when Σ^+ is the full shift, r depends only on one letter, and $\sum_{a\in\mathcal{W}} e^{r(a)} = 1$.

In general we have that $P_{\mathrm{Gur}}(r, T_\psi) \leq \log \operatorname{spr}(\mathcal{L}_r) \leq \log \operatorname{spr}(L_r) = P(r,\sigma)$. If we assume in addition that the pair (ψ, r) is weakly symmetric and that T_ψ is transitive (see Chapters 2 and 3 for the definition) then we have that $P_{\mathrm{Gur}}(r, T_\psi) = \log \operatorname{spr}(\mathcal{L}_r)$ [22]. This should be compared with the random walk hypothesis that the probability is symmetric and has generating support. The question of equality of $\log \operatorname{spr}(\mathcal{L}_r) \leq \log \operatorname{spr}(L_r)$ can be seen to be controlled by properties of the group, as stated now.

Proposition 1.2.1 (Stadlbauer [50]). *Assume $T_\psi : \Sigma^+ \times G \to \Sigma^+ \times G$ is transitive. If G is non-amenable, then* $\operatorname{spr}(\mathcal{L}_r) < \operatorname{spr}(L_r)$. *Assuming that (ψ, r) is weakly symmetric, the converse holds: if G is amenable, then $P_{\mathrm{Gur}}(r, T_\psi) = P(r,\sigma)$*

We study a family of group extensions which can be seen as quotients of a fixed group extension $\psi : \Sigma^+ \to G$. For each $H \trianglelefteq G$, write $\psi_H(x)$ for the coset of G/H given by $\psi(x)$. In this way $T_{\psi_H} : \Sigma^+ \times G/H \to \Sigma^+ \times G/H$ is a group extension with skewing function ψ_H. For notational convenience, write $\mathcal{L}_{r,H}$ for the transfer operator given by r and T_{ψ_H}; and write $\mathcal{C}_H^\infty$ for the Banach space assosciated to $\mathcal{L}_{r,H}$. We also consider a family of transfer operators $\mathcal{L}_{r_s,H}$ where $s \mapsto r_s \in F_\theta$, $s \in [-1,1]$, is continuous in the $\|\cdot\|_\theta$ topology. See Chapter 3 for a precise definition of the Banach space F_θ.

By Proposition 1.2.1, if $H \trianglelefteq G$ with G/H non-amenable and $T_{\psi_H} : \Sigma^+ \times G/H \to \Sigma^+ \times G/H$ is transitive, then $\operatorname{spr}(\mathcal{L}_{r,H}) < \operatorname{spr}(L_r)$. The proof in [50] finds an upper bound for $\operatorname{spr}(\mathcal{L}_{r,H})$ that depends on the first return to a cylinder under T_{ψ_H}. As this bound does not suffice for our needs, we introduce a new condition on ψ that removes this dependency. This condition can be seen as a weakening of transitivity.

Definition 1.2.1. We say that (Σ^+, G, ψ) satisfies *linear visibility with remainder* (LVR) if there exists a map $\chi : G \to \bigcup_{n=1}^\infty \mathcal{W}^n$ with the following properties:

- (visibility with remainder) there exists a finite set $\mathcal{R} \subset G$ such that for every $g \in G$, there are $r_1, r_2 \in \mathcal{R}$ with $\psi^{k_g}(\chi(g)) = r_1 g r_2$, where k_g is the length of the word $\chi(g)$;

- (linear growth) there exists L such that for any finite collection $g_1, \ldots, g_r \in G$, writing $g = g_1 \cdots g_r$, we have that $k_g \leq L(\sum_{i=1}^r k_{g_i})$, where k_g is the length of $\chi(g)$, and k_{g_i} the length of $\chi(g_i)$, for each i.

Theorem 1.2.3. *Let A be a finite generating set for G, and let $\mathcal{N}$ be a collection of normal subgroups of G.*

(i) Assume that (ψ, r) is weakly symmetric. Then

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) = 0 \implies \sup_{H \in \mathcal{N}} P_{\mathrm{Gur}}(r, T_{\psi_H}) = P(r, \sigma).$$

(ii) Assume that (Σ^+, G, ψ) satisfies (LVR). *Then*

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) > 0 \implies \sup_{H \in \mathcal{N}} \mathrm{spr}(\mathcal{L}_{r,H}) < \mathrm{spr}(L_r).$$

(iii) In addition, in case (ii) suppose that $s \mapsto r_s$ is continuous (in the $\|\cdot\|_\theta$ topology) for $s \in [-1, 1]$. Then

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) > 0 \implies \sup_{H \in \mathcal{N}, s \in [-\delta, \delta]} \mathrm{spr}(\mathcal{L}_{r_s,H}) < \mathrm{spr}(L_{r_0}),$$

for some $\delta > 0$.

Chapter 2

Amenability, critical exponents of subgroups and growth of closed geodesics

Let Γ_0 be a (non-elementary) convex cocompact group of isometries of a pinched Hadamard manifold X. We show that a normal subgroup Γ of Γ_0 has critical exponent equal to the critical exponent of Γ_0 if and only if Γ_0/Γ is amenable. We prove a similar result for the exponential growth rate of closed geodesics on X/Γ_0. These statements are analogues of classical results of Kesten for random walks on groups and Brooks for the spectrum of the Laplacian on covers of Riemannian manifolds.

The contents of this chapter are joint work with Richard Sharp. These results are published in Mathematische Annalen [14].

2.1 Introduction

Let X be a connected simply connected and complete Riemannian manifold with sectional curvatures bounded between two negative constants. We call such an X a *pinched Hadamard manifold*. Let Γ_0 be a *non-elementary* and *convex cocompact* group of isometries of X (see section 2 for precise definitions) and let $\Gamma \trianglelefteq \Gamma_0$ be a normal subgroup. We write $G = \Gamma_0/\Gamma$. We define the *critical exponent* of Γ_0 to be the abscissa of convergence of the Poincaré series

$$\sum_{g\in\Gamma_0} e^{-sd_X(x,gx)},$$

for any choice of base point $x \in X$, and denote it by δ_{Γ_0}. The critical exponent δ_Γ of Γ is defined in the same way. The fact that Γ_0 is non-elementary and convex cocompact means that $\delta_{\Gamma_0} > 0$ and it is clear that $\delta_\Gamma \leq \delta_{\Gamma_0}$. It is natural to ask when we have equality and our main result will give a precise answer to this question, which will depend only on G as an abstract group. (We will discuss the history of this and related problems in the next section.)

Before stating our result, we introduce an alternative formulation. Consider the quotient manifolds $M_0 = X/\Gamma_0$ and $M = X/\Gamma$; M is a regular cover of M_0 with covering group G. Then M_0 has a countably infinite set $\mathcal{C}(M_0)$ of closed geodesics (which are not assumed to be prime). For $\gamma \in \mathcal{C}(M_0)$, we write $|\gamma|$ for its length. For each $T > 0$, the set $\{\gamma \in \mathcal{C}(M_0) : |\gamma| \leq T\}$ is finite and we can define a number $h_0 = h(M_0) > 0$ by

$$h(M_0) = \lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{C}(M_0) : |\gamma| \leq T\}.$$

(The limit exists and, in fact, $\lim_{T\to\infty} Te^{-h(M_0)T} \#\{\gamma \in \mathcal{C}(M_0) : |\gamma| \leq T\} = 1/h(M_0)$ [33, 36].) Similarly, we write $\mathcal{C}(M)$ for the set of closed geodesics on M. If G is infinite then, for a given T, the set $\{\gamma \in \mathcal{C}(M) : |\gamma| \leq T\}$ is infinite (since a single closed geodesic has infinitely many images under the action of G). However, we can obtain a finite quantity by choosing any relatively compact open subset W of the unit-tangent bundle SM that intersects the non-wandering set for the geodesic flow and considering the set

$$\mathcal{C}(M, W) = \{\gamma \in \mathcal{C}(M) : \hat{\gamma} \cap W \neq \varnothing\},$$

where $\hat{\gamma}$ is the periodic orbit for the geodesic flow lying over the closed geodesic γ. Following [35], we then define $h(M) \leq h(M_0)$ by

$$h(M) = \limsup_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{C}(M, W) : |\gamma| \leq T\}$$

and this is independent of the choice of W. Again we may ask when we have equality.

It is well known that $h(M_0) = \delta_{\Gamma_0}$, and both are equal to the topological entropy of the geodesic flow over X/Γ_0. Furthermore, the work of Paulin, Pollicott and Schapira in [35] shows that $h(M) = \delta_\Gamma$ (and that the limsups in the definitions are, in fact, limits). We will discuss this further in section 2.

Our main result is the following.

Theorem 2.1.1. *Let Γ_0 be a convex cocompact group of isometries of a pinched Hadamard manifold X and let Γ be a normal subgroup of Γ_0. Then the following* *are equivalent:*

(i) $\delta_\Gamma = \delta_{\Gamma_0}$,

(ii) $h(X/\Gamma) = h(X/\Gamma_0)$,

(iii) $G = \Gamma_0/\Gamma$ *is amenable.*

In view of the equivalence of (i) and (ii) discussed above, this will follow from Theorem 2.7.1 below, in which we prove the equivalence of (ii) and (iii).

Remark 2.1.1. In fact, that (iii) implies (i) is a theorem of Roblin [43] (see also the expository account in [44]), which actually applies in the more general situation where X is a CAT(-1) space.

Remark 2.1.2. In the special case where $X = \boldsymbol{H}^n_{\mathbb{R}}$, n-dimensional real hyperbolic space, and $\delta_{\Gamma_0} > (n-1)/2$, the statement $\delta_\Gamma = \delta_{\Gamma_0}$ if and only if G is amenable is a result of Brooks [7] (and holds when Γ_0 is geometrically finite, which is a more general condition than convex cocompactness). We discuss this in more detail in the next section. If $X = \boldsymbol{H}^n_{\mathbb{R}}$ and Γ_0 is essentially free then Stadlbauer showed the same result holds without the assumption $\delta_{\Gamma_0} > (n-1)/2$ [50]. The class of essentially free groups includes all non-cocompact geometrically finite Fuchsian groups (i.e. discrete groups of isometries of $\boldsymbol{H}^2_{\mathbb{R}}$) and all Schottky groups.

We conclude the introduction by outlining the contents of this chapter. In the next section, we define the key concepts associated to groups that are mentioned above and discuss some history of this and analogous problems. Our approach to Theorem 2.1.1 is via dynamics. More precisely, we consider the geodesic flow over M_0 and M and a class of symbolic dynamical systems that model them. These symbolic systems belong to a class called countable state Markov shifts: we introduce these in section 3 and define a key quantity called the Gurevič pressure. We also mention recent results of Stadlbauer that will be key to our analysis. In sections 4, 5 and 6, we consider the geodesic flows over M_0 and M and discuss how they may be modelled by symbolic systems, particular using a group extension construction to record information about lifts to the cover. Finally, in section 7, we link various zeta functions defined by the closed geodesics to the Gurevič pressure and hence, using Stadlbauer's result, prove that the equality of $h(M_0)$ and $h(M)$ is equivalent to amenability of the covering group.

2.2 Background and history

Let X be a pinched Hadamard manifold, i.e. a connected simply connected complete Riemannian manifold such that its sectional curvatures lie in an interval $[-\kappa_1, -\kappa_2]$, for some $\kappa_1 > \kappa_2 > 0$. Associated to X is a well defined topological space ∂X called the Gromov boundary. This can be defined to be the set of equivalence classes of geodesic rays emanating from a fixed base point, where two rays are equivalent it their distance apart is bounded above. Moreover, there is a natural topology of $X \cup \partial X$ such that the inclusion of X into $X \cup \partial X$ is continuous, and $X \cup \partial X$ is compact. Let Γ_0 be a group of isometries acting freely and properly discontinuously on X. We say that Γ_0 is *non-elementary* if it is not a finite extension of a cyclic group. Fix $x \in X$. Then the orbit $\Gamma_0 x = \{gx : g \in \Gamma_0\}$ accumulates only on ∂X and we call the set of accumulation points $L(\Gamma_0)$ the *limit set* of Γ_0; this is independent of the choice of o. Let $C(\Gamma_0)$ denote the intersection of X with the convex hull (with respect to the metric on X) of $L(\Gamma_0)$. We say that Γ_0 is *convex cocompact* if $C(\Gamma_0)/\Gamma_0$ is compact. If Γ_0 is convex cocompact then $M_0 = X/\Gamma_0$ has a compact core: a manifold with boundary M_0 which contains $\mathcal{C}(M_0)$.

Now suppose that $\Gamma \trianglelefteq \Gamma_0$ and that $M = X/\Gamma$. Then M is a regular $G = \Gamma_0/\Gamma$-cover of M_0. Let $\pi : M \to M_0$ denote the projection. It is shown in [35] that if W is a relatively compact open subset of the unit-tangent bundle SM intersecting the non-wandering set for the geodesic flow $\phi^t : SM \to SM$ for any $c > 0$,

$$\delta_\Gamma = \lim_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{C}(M, W) : T - c < |\gamma| \le T\}.$$

(To see this, take $F = 0$ in Theorem 1.1 of [35].)

Lemma 2.2.1. *We have $h(M) = \delta_\Gamma$.*

Proof. If $\delta_\Gamma > 0$ when we may replace the condition $|\gamma| \le T$ with $T-c < |\gamma| \le T$ without affecting the exponentional growth rate and conclude the result. On the other hand, if $\delta_\Gamma = 0$ then the simple inequality $h(M) \le \delta_\Gamma$ gives $h(M) = 0$. □

Recall from Chapter 1, that a (countable) group G is *amenable* if $\ell^\infty(G)$ admits an invariant mean, i.e. that there exists a bounded linear functional $\nu : \ell^\infty(G) \to \mathbb{R}$ such that, for all $f \in \ell^\infty(G)$,

1. $\inf_{g\in G} f(g) \le \nu(f) \le \sup_{g\in G} f(g)$; and
2. for all $g \in G$, $\nu(f_g) = \nu(f)$, where $f_g(h) = f(g^{-1}h)$.

The concept was introduced by von Neumann in 1929. One sees immediately from this definition that finite groups are amenable by taking

$$\nu(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

An alternative criterion for amenability was given by Følner [16]: G is amenable if and only if, for every $\epsilon > 0$ and every finite set $\{g_1 \ldots, g_n\} \subset G$, there exists a finite set $F \subset G$ such that $\#(F \cap g_i F) \geq (1-\epsilon)\#F$, $i = 1, \ldots, n$. Using this criterion, it is easy to see that abelian groups are amenable and that, more generally, groups with subexponential growth are amenable [20]. Furthermore, since amenability is closed under extensions, solvable groups are amenable. In particular, there are amenable groups with exponential growth (e.g. lamplighter groups). On the other hand, a group containing the free group on two generators is not amenable and non-elementary Gromov hyperbolic groups (a class which includes the convex cocompact groups above) are not amenable.

There are numerous results that connect growth and spectral properties of groups and manifolds to amenability. The prototype is the following theorem of Kesten from 1959. Let G be a countable group and let $p : G \to \mathbb{R}_{\geq 0}$ be a symmetric probability distribution (i.e. $\sum_{g \in G} p(g) = 1$ and $p(g^{-1}) = p(g)$ for all $g \in G$) such that its support, $\mathrm{supp}(p)$, generates G. This defines a symmetric random walk on G with transition probabilities $P(g, g') = p(g^{-1}g')$. If we define $\lambda(G, P)$ to be the $\ell^2(G)$-spectral radius of P then we have

$$\lambda(G, P) = \limsup_{n \to \infty} P^n(g, g)^{1/n} = \lim_{n \to \infty} P^{2n}(g, g)^{1/2n},$$

for any $g \in G$. It is clear that $\lambda(G, P) \leq 1$.

Theorem 2.2.1 (Kesten [25])**.** *We have $\lambda(G, P) = 1$ if and only if G is amenable.*

Note that, while $\lambda(G, P)$ depends on the probability p, whether or not it takes the value 1 depends only on the group G.

Subsequently, results inspired by Kesten's Theorem were obtained in a variety of other situations. In the setting of group theory, the most notable result is Grigorchuk's co-growth criterion [18] (see also Cohen [10]) for finitely generated groups. Recall that a finitely generated group G may be written as F/N, where F is a free group of rank k and N is a normal subgroup. If $|\cdot|$ denotes the word length on F with respect to a free generating set then $\lim_{n \to \infty}(\#\{x \in F : |x| = n\})^{1/n} = 2k - 1$.

Theorem 2.2.2 (Grigorchuk [18])**.** *We have*

$$\limsup_{n\to\infty} (\#\{x \in N : |x| = n\})^{1/n} = 2k - 1$$

if and only if G is amenable.

Subsequently, various extensions of this to graphs and (non-backtracking) random walks were obtained by Woess [55], Northshield [29,30] and Ortner and Woess [32].

In the setting of Riemannian manifolds, an analogue is provided by the following spectral result of Brooks. Let M_0 be a complete Riemannian manifold which is of "finite topological type", i.e. that it is topologically the union of finitely many simplices, and let M be a regular covering of M with covering group G. Let $\lambda_0(M_0)$ and $\lambda_0(M)$ denote the infimum of the spectrum of the Laplace-Beltrami operator on M_0 and M, respectively; then $\lambda_0(M) \geq \lambda_0(M_0)$. Brooks showed that amenability of G implied equality and that, together with an additional condition, the converse holds. More precisely, he proved the following.

Theorem 2.2.3 (Brooks [7])**.** *(i) If G is amenable then $\lambda_0(M) = \lambda_0(M_0)$.*
(ii) Let ϕ be the lift of a $\lambda_0(M_0)$-harmonic function to M and let F be a fundamental region for M_0 on M. Suppose that there is a compact $K \subset F$ such that

$$\inf_S \frac{\int_S \phi^2 \, d\mathrm{Area}}{\int_{\mathrm{int}(S)} \phi^2 \, d\mathrm{Vol}} > 0,$$

where the infimum is taken over co-dimension 1 submanifolds S that divide $F \backslash K$ into an interior and an exterior. If $\lambda_0(M) = \lambda_0(M_0)$ then G is amenable.

(See also Brooks [6] for the case when M_0 is compact and G is its fundamental group and Burger [8] for a shorter proof.) The Cheeger-type condition in part (ii) holds if, for example, M_0 is a convex cocompact quotient of the n-dimensional real hyperbolic space $\boldsymbol{H}^n_{\mathbb{R}}$ and $\lambda_0(M_0) < (n-1)^2/4$.

The problem of critical exponents was also first considered in the case $X = \boldsymbol{H}^n_{\mathbb{R}}$. In the early 1980s, Rees [42] showed that if Γ_0/Γ is abelian then we have equality (and her dynamical methods generalize to variable curvature). Soon afterwards, Brooks obtained a more general result as a corollary of Theorem 2.2.3. This is due to the fact that, for $X = \boldsymbol{H}^n_{\mathbb{R}}$, δ_{Γ_0} and $\lambda_0(M_0)$ are related by the formula $\lambda_0(M_0) = \delta_{\Gamma_0}(n - 1 - \delta_{\Gamma_0})$, provided $\delta_{\Gamma_0} > (n-1)/2$, with the same holding for δ_Γ and $\lambda_0(M)$. In particular, if $\delta_{\Gamma_0} > (n-1)/2$ then $\lambda_0(M) < (n-1)^2/4$ and so Theorem 2.2.3 implies the statement that $\delta_\Gamma = \delta_{\Gamma_0}$ if and only if G is amenable.

More recently, Stadlbauer [50] and Jaerisch [22] have considered the relation between amenability and certain growth rates that occur in the study of group extensions of dynamical systems. It will be clear below that we are greatly indebted to this work in our analysis.

2.3 Countable state Markov shifts and Gurevič pressure

In this subsection we will define countable state Markov shifts and discuss some of their properties. Basic definitions and results are taken from chapter 7 of [26]. In the rest of the paper, we will be concerned with finite state shifts and group extensions of these by a countable group, so we shall often specialise to these two cases.

Let S be a countable set, called the *alphabet*, and let A be a matrix, called the transition matrix, indexed by $S \times S$ with entries zero or one. We then define the space

$$\Sigma_A^+ = \left\{ x = (x_n)_{n=0}^{\infty} \in S^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}^+ \right\},$$

with the product topology induced by the discrete topology on S. This topology is compatible with the metric $d(x,y) = 2^{-n(x,y)}$, where

$$n(x,y) = \inf\{n : x_n \neq y_n\},$$

with $n(x,y) = \infty$ if $x = y$. If S is finite then Σ_A^+ is compact. We say that A is *locally finite* if all its row and column sums are finite. Then Σ_A^+ is locally compact if and only if A is locally finite. (The group extensions we consider have this latter property.)

We define the (one-sided) countable state topological Markov shift $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by $(\sigma x)_n = x_{n+1}$. This is a continuous map. We will say that σ is *topologically transitive* if it has a dense orbit and *topologically mixing* if, given non-empty open sets $U, V \subset \Sigma_A^+$, there exists $N \geq 0$ such that $\sigma^{-n}(U) \cap V \neq \varnothing$ for all $n \geq N$. We say that the matrix A is *irreducible* if, for each $(i,j) \in S \times S$, there exists $n = n(i,j) \geq 1$ such that $A^n(i,j) > 0$. For A irreducible, set $p \geq 1$ to be the greatest common divisor of periods of periodic orbits $\sigma : \Sigma_A^+ \to \Sigma_A^+$; this p is called the period of A. We say that A is *aperiodic* if $p = 1$ or, equivalently, if there exists $n \geq 1$ such that A^n has all entries positive. Suppose that A is locally finite. Then $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically transitive if and only if A is irreducible and $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically mixing if and only if

A is aperiodic.

Suppose that A is irreducible but not aperiodic and fix $i \in S$. As in [26], we may partition S into sets S_l, $l = 0, \ldots, p-1$, defined by

$$S_l = \{j : A^{np+l}(i,j) > 0 \text{ for some } n \geq 1\}.$$

(This partition is independent of the choice of i.) For each l, let A_l denote the restriction of A to $S_l \times S_l$; then $\sigma : \Sigma_{A_l}^+ \to \Sigma_{A_{l+1}}^+$ (mod p) and A_l^p is aperiodic.

We say that an n-tuple $w = (w_0, \ldots, w_{n-1}) \in S^n$ is an *allowed word* of length n if $A(w_j, w_{j+1}) = 1$ for $j = 0, \ldots, n-2$. We will write $\mathcal{W}^n$ for the set of allowed words of length n. If $w \in \mathcal{W}^n$ then we define the associated cylinder set $[w]$ by

$$[w] = \{x \in \Sigma_A^+ : x_j = w_j,\ j = 0, \ldots, n-1\}.$$

For a function $f : \Sigma_A^+ \to \mathbb{R}$, set

$$V_n(f) = \sup\{|f(x) - f(y)| : x_j = y_j,\ j = 0, \ldots, n-1\}.$$

We say that f is *locally Hölder continuous* if there exist $0 < \theta < 1$ and $C \geq 0$ such that, for all $n \geq 1$, $V_n(f) \leq C\theta^n$. (There is no requirement of $V_0(f)$ and a locally Hölder f may be unbounded.) For $n \geq 1$, we write

$$f^n := f + f \circ \sigma + \cdots + f \circ \sigma^{n-1}.$$

Definition 2.3.1. Suppose that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically transitive and let $f : \Sigma_A^+ \to \mathbb{R}$ be a locally Hölder continuous function. Following Sarig [46], we define the *Gurevič pressure*, $P_{\mathrm{Gur}}(f, \sigma)$, of f to be

$$P_{\mathrm{Gur}}(f, \sigma) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ x_0 = a}} e^{f^n(x)},$$

where $a \in S$. (The definition is independent of the choice of a.)

Remark 2.3.1. In [46], Sarig gives this definition in the case where $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically mixing. However, the above decomposition of $\Sigma_A^+ = \Sigma_{A_0}^+ \cup \cdots \cup \Sigma_{A_{p-1}}^+$, with σ^p topologically mixing on each component, together with the regularity of the function f, shows that the same definition may be made in the topologically transitive case.

We now specialise to the case where S is finite. In this situation, we call $\sigma : \Sigma_A^+ \to \Sigma_A^+$ a (one-sided) subshift of finite type. The above definitions and results hold. If $f : \Sigma_A^+ \to \mathbb{R}$ is Hölder continuous then f is locally Hölder. Provided $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically transitive, the Gurevič pressure $P_{\mathrm{Gur}}(f,\sigma)$ agrees with the standard pressure $P(f,\sigma)$, defined by

$$P(f,\sigma) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} e^{f^n(x)}$$

and if σ is topologically mixing then the lim sup may be replaced with a limit.

We now consider *group extensions*, or in some literature *skew products*, of a shift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$, which we will assume to be topologically mixing. Let G be a countable group and let $\psi : \Sigma_A^+ \to G$ be a function depending only on two co-ordinates, $\psi(x) = \psi(x_0, x_1)$. (One may consider more general ψ but this set-up suffices for our needs.) This data defines a *group extension* $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ by $T_\psi(x,g) = (\sigma x, g\psi(x)^{-1})$. For $n \geq 1$ define ψ_n by

$$\psi_n(x) = \psi(\sigma^{n-1}x) \cdots \psi(\sigma x)\psi(x).$$

(In Chapter 3, we make use of a different function $\psi^n(x) = \psi(x)\psi(\sigma x) \cdots \psi(\sigma^{n-1}x)$. These are related by $\psi^n(x) = e$ if and only if $\psi_n(x) = e$, where e is the identity element in G.) Then $T_\psi^n(x,g) = (x,g)$ if and only if $\sigma^n x = x$ and $\psi_n(x) = e$.

The map $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ is itself a countable state Markov shift with alphabet $S \times G$ and transition matrix $\widetilde{A}$ defined by $\widetilde{A}((i,g),(j,h)) = 1$ if $A(i,j) = 1$ and $\psi(i,j) = h^{-1}g$ and $\widetilde{A}((i,g),(j,h)) = 0$ otherwise. Clearly, $\widetilde{A}$ is locally finite and so the topological transitivity and topological mixing of T_ψ are equivalent to $\widetilde{A}$ being irreducible and aperiodic, respectively.

Let $f : \Sigma_A^+ \to \mathbb{R}$ be Hölder continuous and define $\widetilde{f} : \Sigma_A^+ \times G \to \mathbb{R}$ by $\widetilde{f}(x,g) = f(x)$; then $\widetilde{f}$ is locally Hölder continuous and its Gurevič pressure $P_{\mathrm{Gur}}(\widetilde{f}, T_\psi)$ is defined. In fact, it is easy to see that, due to the mixing of σ,

$$P_{\mathrm{Gur}}(\widetilde{f}, T_\psi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} e^{f^n(x)}.$$

It is clear that $P_{\mathrm{Gur}}(\widetilde{f}, T_\psi) \leq P(f,\sigma)$ and it is interesting to ask when equality holds. Stadlbauer has shown this depends only on the amenability of the group G, provided the group extension and the function f satisfy appropriate symmetry conditions, which we now describe.

Suppose there is a fixed point free involution $\kappa : S \to S$ such that $A(\kappa j, \kappa i) = A(i,j)$, for all $i,j \in S$. We say that the group extension $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ is *symmetric* (with respect to κ) if $\psi(\kappa j, \kappa i) = \psi(i,j)^{-1}$. A function $f : \Sigma_A^+ \to \mathbb{R}$ is called *weakly symmetric* if, for all $n \geq 1$ and and all length n

cylinders $[z_0, z_1, \ldots, z_{n-1}]$, there exists $D_n > 0$ such that $\lim_{n\to\infty} D_n^{1/n} = 1$ and

$$\sup_{\substack{x\in[z_0,\ldots,z_{n-1}]\\ y\in[\kappa z_{n-1},\ldots,\kappa z_0]}} \exp(f^n(x) - f^n(y)) \leq D_n.$$

The following is the main result of Stadlbauer [50], restricted to the case where the base is a (finite state) subshift of finite type. We will use this in subsequent arguments. (More generally, Stadlbauer considers group extensions of countable state Markov shifts.)

Proposition 2.3.1 (Stadlbauer [50], Theorem 4.1 and Theorem 5.6)**.** *Let $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ be a transitive, symmetric group extension of a mixing subshift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by a countable group G. Let $f : \Sigma_A^+ \to \mathbb{R}$ be a weakly symmetric Hölder continuous function and define $\widetilde{f} : \Sigma_A^+ \times G \to \mathbb{R}$ by $\widetilde{f}(x, g) = f(x)$. Then $P_{\mathrm{Gur}}(\widetilde{f}, T_\psi) = P(f, \sigma)$ if and only if G is amenable.*

Remark 2.3.2. In [50], Stadlbauer considers group extensions with ψ depending on only one coordinate. However, replacing S by $\mathcal{W}^2$, one can easily recover the above formulation.

We end this section by discussing two-sided subshifts of finite type and suspended flows over them. Given a finite alphabet S and transition matrix A, we define

$$\Sigma_A = \left\{ x = (x_n)_{n=0}^\infty \in S^{\mathbb{Z}} \,:\, A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z} \right\}$$

and the (two-sided) shift of finite type $\sigma : \Sigma_A \to \Sigma_A$ by $(\sigma x)_n = x_{n+1}$. As before, we give Σ_A with the product topology induced by the discrete topology on S and this is compatible with the metric $d(x, y) = 2^{-n(x,y)}$, where

$$n(x, y) = \inf\{|n| \,:\, x_n \neq y_n\},$$

with $n(x, y) = \infty$ if $x = y$. Then Σ_A is compact and σ is a homeomorphism. There is an obvious one-to-one correspondence between the periodic points of $\sigma : \Sigma_A \to \Sigma_A$ and $\sigma : \Sigma_A^+ \to \Sigma_A^+$. Furthermore, we may pass from Hölder functions on Σ_A to Hölder functions on Σ_A^+ in such a way that sums around periodic orbits are preserved. More precisely, there is the following lemma, due originally to Sinai. (See, for instance [33].)

Lemma 2.3.1. *Let $f : \Sigma_A \to \mathbb{R}$ be Hölder continuous. Then there is a Hölder continuous function $f' : \Sigma_A^+ \to \mathbb{R}$ (with a smaller Hölder exponent) such that $f^n(x) = (f')^n(x)$, whenever $\sigma^n x = x$.*

We may also define suspended flows over $\sigma : \Sigma_A \to \Sigma_A$. Given a strictly positive continuous function $r : \Sigma_A \to \mathbb{R}^+$, we define the r-suspension space

$$\Sigma_A^r = \{(x, s) : x \in \Sigma_A, \ 0 \le s \le r(x)\}/ \sim,$$

where $(x, r(x)) \sim (\sigma x, 0)$. The suspended flow $\sigma_r^t : X_A^r \to X_A^r$ is defined by $\sigma_r^t(x, s) = (x, s + t)$ modulo the identifications. Clearly, there is a natural one-to-one correspondence between periodic orbits for $\sigma_r^t : \Sigma_A^r \to \Sigma_A^r$ and periodic orbits for $\sigma : \Sigma_A \to \Sigma_A$, and a σ_r-periodic orbit is prime if and only if the corresponding σ-periodic orbit is prime. Furthermore, if γ is a closed σ_r-orbit corresponding to the closed σ-orbit $\{x, \sigma x, \dots, \sigma^{n-1}x\}$ then the period of γ is equal to $r^n(x)$.

2.4 Coverings and geodesic flows

As in the introduction, we shall write $M_0 = X/\Gamma_0$, $M = X/\Gamma$ and $G = \Gamma_0/\Gamma$. There is a natural dynamical system related to the geometry of M_0, namely the geodesic flow on the unit-tangent bundle $SM_0 = \{(x, v) \in TM_0 : \|v\|_x = 1\}$, where $\| \cdot \|_x$ is the norm induced by the Riemannian structure on $T_x M_0$. For future reference, we write $p : SM_0 \to M_0$ for the projection. The geodesic flow $\phi_0^t : SM_0 \to SM_0$ is defined as follows. Given $(x, v) \in SM_0$, there is a unique unit-speed geodesic $\gamma : \mathbb{R} \to M_0$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. We then define $\phi_0^t(x, v) = (\gamma(t), \dot{\gamma}(t))$.

The non-wandering set $\Omega(\phi_0) \subset SM_0$ is defined to be the set of points $x \in SM_0$ with the property that for every open neighbourhood U of x, there exists $t > 0$ such that $\phi_0^t(U) \cap U \neq \varnothing$. It can be characterised as the set of vectors tangent to $C(\Gamma_0)/\Gamma_0 \subset M_0$.

The restriction of the geodesic flow to its non-wandering set, $\phi_0^t : \Omega(\phi_0) \to \Omega(\phi_0)$, is an example of a *hyperbolic flow*. A C^1 flow $f^t : \Omega \to \Omega$ is hyperbolic if

1. there is a continuous Df-invariant splitting of the tangent bundle

$$T_\Omega(SM) = E^0 \oplus E^s \oplus E^u,$$

where E^0 is the line bundle tangent to the flow and where there exists constants $C, c > 0$ such that

(i) $\|Df^t v\| \le Ce^{-ct}\|v\|$, for all $v \in E^s$ and $t > 0$;

(ii) $\|Df^{-t} v\| \le Ce^{-ct}\|v\|$, for all $v \in E^u$ and $t > 0$,

2. $f^t : \Omega \to \Omega$ is transitive (i.e. it has a dense orbit),

3. the periodic f-orbits are dense in Ω, and

4. there is an open set $U \supset \Omega$ such that $\Omega = \bigcap_{t \in \mathbb{R}} f^t(U)$.

The manifold M is a regular G-cover of M_0 and we let $\pi : M \to M_0$ denote the covering map. The geodesic flow $\phi^t : SM \to SM$ is defined in a similar way to the geodesic flow on SM_0. We also write $p : SM \to M$ for the projection. The action of G extends to the unit-tangent bundle SM by the formula $g(x, v) = (gx, Dg_x v)$ and induces a regular covering $\pi : SM \to SM_0$ which commutes with the two flows. (The use of π to denote both coverings should not cause any confusion.)

There is a natural one-to-one correspondence between (prime) periodic orbits for $\phi_0^t : \Omega(\phi_0) \to \Omega(\phi_0)$ and (prime) closed geodesics on M_0, with the least period being equal to the length of the closed geodesic. We will typically write γ for either a closed geodesic or a periodic orbit and allow the context to distinguish them. We will write $|\gamma|$ for the length (period) of γ. The number $h_0 = h(M_0)$ defined in the introduction as the exponential growth rate of the number of γ with $|\gamma| \leq T$ is also equal to the topological entropy $h_{\mathrm{top}}(\phi_0)$ of ϕ_0.

2.5 Markov sections and symbolic dynamics

A particularly useful aspect of hyperbolic flows is that they admit a description by finite state symbolic dynamics. We shall outline this construction below.

Given $\epsilon > 0$, we define the (strong) *local stable manifold* $W_\epsilon^s(x)$ and (strong) *local unstable manifold* $W_\epsilon^u(x)$ for a point $x \in SM_0$ by

$$W_\epsilon^s(x) = \left\{ y \in SM_0 : \sup_{t \geq 0} d(\phi_0^t(x), \phi_0^t(y)) \leq \epsilon, \lim_{t \to \infty} d(\phi_0^t(x), \phi_0^t(y)) = 0 \right\}$$

and

$$W_\epsilon^u(x) = \left\{ y \in SM_0 : \sup_{t \geq 0} d(\phi_0^{-t}(x), \phi_0^{-t}(y)) \leq \epsilon, \lim_{t \to \infty} d(\phi_0^{-t}(x), \phi_0^{-t}(y)) = 0 \right\}.$$

Provided $\epsilon > 0$ is sufficiently small, these sets are diffeomorphic to $(\dim SM_0 - 1)$-dimensional embedded disks. If x and y are sufficiently close then there is a unique $t \in [-\epsilon, \epsilon]$ such that $W_\epsilon^s(x) \cap W_\epsilon^u(\phi_0^t(y)) \neq \varnothing$ and, furthermore, this intersection consists of a single point denoted $[x, y]$. This pairing $[\cdot, \cdot]$ is called the *local product structure*.

Let $D_0^1, \ldots, D_0^k$ be a family of co-dimension one disks that form a local cross section to the flow and let P_0 denote the Poincaré map between them. For each $i = 1, \ldots, k$, let $S_0^i \subset \mathrm{int}(D_0^i) \cap \Omega(\phi_0)$ be sets which are chosen to be *rectangles* in the sense that whenever $x, y \in S_0^i$ then $[x, y] \in S_0^i$ and *proper* (i.e. $S_0^i = \overline{\mathrm{int}(S_0^i)}$ for each i). (Here and subsequently, the interiors are taken relative to D_0^i.) We then say that $S_0^1, \ldots, T = S_0^k$ are *Markov sections* for the flow if

1. for $x \in \mathrm{int}(S_0^i)$ with $P_0 x \in \mathrm{int}(S_0^j)$ then $P_0(W^s(x, S_0^i)) \subset W^s(P_0 x, S_0^j)$, and
2. for $x \in \mathrm{int}(S_0^i)$ with $P_0^{-1} x \in \mathrm{int}(S_0^j)$ then $P_0^{-1}(W^u(x, S_0^i)) \subset W^u(P_0^{-1} x, S_0^j)$,

where $W^s(x, S_0^i)$ and $W^u(x, S_0^i)$ denote the projections of $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ onto $\mathrm{int}(S_0^i)$, respectively.

The local product structure on SM_0 induces a local product structure, also denoted $[\cdot, \cdot]$ on transverse sections by projecting along flow lines. The rectangles T_i may be chosen so that $S_0^i = [U_0^i, V_0^i]$, where U_0^i and V_0^i are closed subsets of local unstable and stable manifolds, respectively. Associated to this, we have projection maps $\rho_i^u : S_0^i \to U_0^i$ and $\rho_i^s : S_0^i \to V_0^i$.

Proposition 2.5.1 (Bowen [5]). *For all $\epsilon > 0$, the flow has Markov sections $S_0^1, \ldots, S_0^k$ such that $\mathrm{diam}(S_0^i) < \epsilon$, for $i = 1, \ldots, k$ and such that $\bigcup_{i=1}^k \phi_0^{[0,\epsilon]} S_0^i = \Omega(\phi_0)$.*

These sections may be chosen to reflect the time-reversal symmetry of the geodesic flow.

Lemma 2.5.1 (Adachi [1], Rees [42]). *The Markov sections $S_0^1, \ldots, S_0^k$ may be chosen so that there is a fixed point free involution $\kappa : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that $A(\kappa j, \kappa i) = 1$ if and only if $A(i, j) = 1$. Furthermore, the involution is consistent with the time reversing involution: $S_0^{\kappa i} = \iota(S_0^i)$, where $\iota : SM_0 \to SM_0$ is the map $\iota(x, v) = (x, -v)$.*

The Markov sections allow us to relate $\phi_0^t : \Omega(\phi_0) \to \Omega(\phi_0)$ to a suspended flow over a mixing subshift of finite type, whose alphabet $\{1, \ldots, k\}$ corresponds to the Markov sections $\{S_0^1, \ldots, S_0^k\}$.

Proposition 2.5.2 (Bowen [5]). *There exists a mixing subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$, a strictly positive Hölder continuous function $r : \Sigma_A \to \mathbb{R}^+$ and a map $\vartheta : \Sigma_A^r \to \Omega(\phi_0)$ such that*

1. *ϑ is a semi-conjugacy (i.e. $\phi_0^t \circ \vartheta = \vartheta \circ \sigma_r^t$);*

2. *ϑ is a surjection and is one-to-one on a residual set;*

3. $h_0 = h_{\mathrm{top}}(\phi_0) = h_{\mathrm{top}}(\sigma_r)$.

Clearly, the fixed point free involution $\kappa : \{1, \ldots, k\} \to \{1, \ldots, k\}$ induces a fixed point free involution $\overline{\kappa} : \Sigma_A \to \Sigma_A$, defined by $(\overline{\kappa}x)_n = \kappa x_{-n}$. Furthermore, $\vartheta \circ \overline{\kappa} = \iota \circ \vartheta$.

The above coding does not give a one-to-one correspondence between periodic orbits for ϕ_0 and σ_r. This is overcome by the following result, which is Bowen's generalisation to flows of a result of Manning for diffeomorphisms [28]. For a flow ξ^t, let $\nu(\xi, T)$ denote the number of prime periodic ξ-orbits of period T and $N_\xi(T)$ the number of periodic ξ-orbits to period at most T.

Lemma 2.5.2 (Bowen [5])**.** *There exist a finite number of subshifts of finite type $\sigma_j : \Sigma_j \to \Sigma_j$ and strictly positive Hölder continuous functions $r_j : \Sigma_j \to \mathbb{R}^+$, $j = 1, \ldots, q$, such that*

1. $h_{\mathrm{top}}(\sigma_{r_j}) < h_{\mathrm{top}}(\phi_0)$, $j = 1, \ldots, q$;

2. $$\nu(\phi_0, T) = \nu(\sigma_r, T) + \sum_{j=1}^{q} (-1)^{\eta_j} \nu(\sigma_{r_j}^i, T),$$

   where $\eta_j \in \{0, 1\}$, $j = 1, \ldots, q$.

Corollary 2.5.1. $N_{\phi_0}(T) = N_{\sigma_r}(T) + O(e^{h'T})$, *where* $h' := \max_{1 \le j \le q} h_{\mathrm{top}}(\sigma_{r_j}) < h_0$.

Finally, we note the following result, which can be found in, for instance, [33].

Lemma 2.5.3. *The entropy h_0 is the unique real number for which $P(-h_0 r, \sigma) = 0$.*

2.6 The group extension

In this section we will describe a group extension of (the one-sided version of) the shift of finite type introduced above, which will serve to encode information about how orbits on SM_0 lift to SM, and relate this construction to the result of Stadlbauer, Proposition 2.3.1, stated above.

Choose $\epsilon_0 > 0$ sufficiently small that every open ball in SM_0 with diameter less than ϵ_0 is simply connected. Let $U_0 \subset SM_0$ be such an open ball. Then $\pi^{-1}(U_0) = \bigcup_{g \in G} gU$, where U is a connected component of $\pi^{-1}(U)$. Since we can choose the Markov sections S_0^i to have arbitrarily small diameters, for each $i = 1, \dots, k$, we can choose an open ball U_0^i of diameter less than ϵ_0 containing S_0^i. As above, we may write $\pi^{-1}(U_0^i) = \bigcup_{g \in G} gU^i$ and $\pi^{-1}(S_0^i) = \bigcup_{g \in G} gS^i$, where $S^i = \pi^{-1}(S_0^i) \cap U^i$, and we may assume that this decomposition is chosen with $\iota(S^i) = S^{\kappa i}$, where ι is the direction-reversing involution $\iota : SM \to SM$ given by $\iota(x, v) = (x, -v)$.

We will use the notation

$$\mathcal{T} = \bigcup_{i=1}^{k} \bigcup_{g \in G} \mathrm{int}(gS^i).$$

(Here and subsequently, the interiors are taken relative to D_0^i.) Notice that each lifted section gS^i is transverse to the flow $\phi^t : SM \to SM$. We write $P : \mathcal{T} \to \mathcal{T}$ for the Poincaré map.

Lemma 2.6.1. *Suppose that $A(i, j) = 1$. There is a unique $g = g(i, j) \in G$ such that for any $x_0 \in S_0^i \cap P_0^{-1}(S_0^j)$, any $x \in \pi^{-1}(x_0)$, and any $h \in G$, if $x \in hS^i$ then $P(x) \in hgS^j$. In addition, $g(\kappa j, \kappa i) = g(i, j)^{-1}$.*

Proof. We will begin be proving the existence and uniqueness of g. Let $x_1, x_2 \in S_0^i \cap P_0^{-1}(S_0^j)$ and let c_1 and c_2 be the ϕ_0-orbit segments from $c_1(0) = x_1$ and $c_2(0) = x_2$ to $c_1(1) = P_0(x_1)$ and $c_2(1) = P_0(x_2)$. We will show that there is a unique $g \in G$ such that the unique lifts of c_1 and c_2 that begin in S^i both have terminal points in gS^j. The statement for all $h \in G$ will follow by translating by the isometry $h \in G$.

Let $\widetilde{c}_1$ and $\widetilde{c}_2$ be lifts of c_1 and c_2 with $\widetilde{c}_1(0), \widetilde{c}_2(0) \in S^i$. Having chosen the Markov partition to have sufficiently small diameters and the flow times between rectangles to be sufficiently small, there is an open ball $U_0 \subset SM_0$ of diameter less than ϵ_0 containing S_0^i, S_0^j, c_1 and c_2. Let U be the connected component of $\pi^{-1}(U_0)$ containing $\widetilde{c}_1$. Then $U \cap S^i \neq \varnothing$ and $U \cap gS^j \neq \varnothing$ and so $S^i \cup gS^j \subseteq U$. It follows that $\widetilde{c}_2$ is entirely contained in U and so we must have $\widetilde{c}_2(1) \in gS^j$ as required.

For the final part, we note that $\iota(\widetilde{c}_1)$ is an orbit segment from $gS^{\kappa j}$ to $S^{\kappa i}$. It follows from the uniqueness in the previous that $g(\kappa j, \kappa i) = g(i, j)^{-1}$. □

We use the preceding lemma to define a group extension of the one-sided shift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$. Define $\psi : \Sigma_A^+ \to G$ (depending on two co-ordinates) by $\psi(x) = \psi(x_0, x_1) = g(x_0, x_1)^{-1}$, where $g = g(x_0, x_1)$ is the unique element of G given by Lemma 2.6.1. Then the group extension

$T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ is defined by

$$T_\psi(x, g) = (\sigma x, g\psi(x)^{-1}).$$

Furthermore, part (2) of Lemma 2.6.1 shows that the group extension is *symmetric* (with respect to the involution κ), i.e. that $\psi(\kappa j, \kappa i) = \psi(i, j)^{-1}$.

We note the relationship between periodic orbits for the lifted geodesic flow and for the group extension.

Lemma 2.6.2. *A periodic ϕ_0-orbit γ in SM_0, corresponding to a periodic σ-orbit $\tau = \{x, \sigma x, \ldots, \sigma^{n-1}x\}$, lifts to a periodic orbit in SM if and only if $\psi_n(x) = e$.*

Proof. We will treat $\vartheta(x)$ as the initial point on γ. Let $\widetilde{\gamma}$ be the lift of γ which starts in S^{x_0}. By Lemma 2.6.1, $\widetilde{\gamma}$ ends in $\psi_n(x)S^{x_0}$ and is thus periodic if and only if $\psi_n(x) = e$. $\square$

We shall apply Proposition 2.3.1 to the group extension $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$. To do this, we need to establish that two further conditions are satisfied: that T_ψ is transitive and that r is weakly symmetric. We start with transitivity.

Lemma 2.6.3. *If G is not equal to $\pi_1(M)$ then the map $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ is transitive.*

Proof. If G is not equal to $\pi_1(M_0)$ then the geodesic flow $\phi^t : SM \to SM$ is transitive. A proof is given in [12] (page 94) for the case where $X = \boldsymbol{H}^2_{\mathbb{R}}$ but the argument clearly generalizes. (See also [15] for the case of variable curvature when Γ_0 is cocompact.) Let $x \in SM$ be a point with dense ϕ-orbit. Without loss of generality $x \in \mathcal{T}$ and then $\{P^n x\}_{n=-\infty}^{\infty}$ is dense in $\mathcal{T}$. Suppose that $\widetilde{A}((i_j, g_j), (i_{j+1}, g_{j+1})) = 1$, where $\widetilde{A}$ is the transition matrix for $\Sigma_A^+ \times G$, for $j = 0, \ldots, n$. Then

$$U = \bigcap_{j=0}^{n} P^{-j}(\mathrm{int}(g_j S^{i_j}))$$

is non-empty and open in $\mathcal{T}$. Since x has dense P-orbit, $P^m x \in U$ for some $m \in \mathbb{Z}$. Then $P^{m+j}(x) \in \mathrm{int}(g_j S^{i_j})$ for $j = 0, \ldots, n$. By definition, this implies that the T_ψ-orbit of $(\vartheta(\pi(x)), g_0) \in \Sigma_A^+ \times G$ (where $\vartheta(\pi(x))$ is identified with a point in the one-sided shift) passes through the (arbitrary) cylinder $[(i_0, g_0), \ldots, (i_n, g_n)]$ and is thus dense in $\Sigma_A^+ \times G$. Therefore, $T_\psi : \Sigma_A^+ \times G \to \Sigma_A^+ \times G$ is transitive. $\square$

Let $r : \Sigma_A \to \mathbb{R}$ be the Hölder continuous function defined by Proposition 2.5.2. By Lemma 2.3.1, there is a Hölder continuous function on Σ_A^+, which we will abuse notation by continuing to call r, with the same sums around periodic orbits.

Lemma 2.6.4. *For any $\xi \in \mathbb{R}$, the function $-\xi r : \Sigma_A^+ \to \mathbb{R}$ is weakly symmetric.*

Proof. It suffices to show that r is weakly symmetric. Since $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is mixing, there exists $N \geq 0$ such that, for each length n cylinder $[\underline{z}] = [z_0, \ldots, z_{n-1}]$, we may find a periodic point $x \in [\underline{z}]$ of period $n+N$. Writing $x = (x_0, \ldots, x_{n+N-1}, x_0, \ldots)$, we set $\kappa x = (\kappa x_{n+N-1}, \ldots, \kappa x_0, \kappa x_{n+N-1}, \ldots)$. Clearly, $\sigma^N(\kappa x)$ is a periodic point of period $n+N$ and $\sigma^N(\kappa x) \in [\kappa \underline{z}]$. Furthermore, $r^{n+N}(x) = |\gamma|$, for some ϕ_0-periodic orbit γ and

$$r^{n+N}(\sigma^N(\kappa x)) = r^{n+N}(\kappa x) = |\iota\gamma| = |\gamma|,$$

where $\iota\gamma$ is the time-reversed periodic orbit corresponding to γ. We therefore have

$$\begin{aligned}\exp(r^n(x) - r^n(\sigma^N(\kappa x))) &= \exp((|\gamma| - r^N(\sigma^n x)) - (|\gamma| - r^N(\sigma^{n+N}(\kappa x))))\\ &= \exp(r^N(\sigma^{n+N}(\kappa x)) - r^N(\sigma^n x)) \leq \exp(2N\|r\|_\infty),\end{aligned}$$

for some constant $C > 0$.

Now let $x' \in [\underline{z}]$ and $y' \in [\kappa\underline{z}]$ be arbitrary. We have

$$\begin{aligned}\exp(r^n(x') - r^n(y')) &= \exp(r^n(x) - r^n(\sigma^N(\kappa x)))\frac{\exp(r^n(x') - r^n(x))}{\exp(r^n(y') - r^n(\sigma^N(\kappa x)))}\\ &\leq \exp(NC)\exp(2c/(1-2^{-\alpha})),\end{aligned}$$

where r satisfies the Hölder condition $|r(x) - r(y)| \leq c d(x,y)^\alpha$. This completes the proof. □

2.7 Zeta functions

In this section we shall prove that the equality of $h_0 = h(M_0)$ and $h(M)$ is equivalent to amenability of G. To do this, we need to relate the growth of closed geodesics in $\mathcal{C}(M, W)$ or, equivalently, of periodic ϕ-orbits which intersect W, to the Gurevič pressure. To do this, we make a particular choice of W, setting $W = \bigcup_{i=1}^k \mathrm{int}(R^i)$, where R^i is the thickened Markov section

$$R^i = \{\phi^t(x) : x \in S^i,\ 0 \leq t \leq \epsilon\}$$

and $0 < \epsilon \leq \inf r$. We now define a zeta function, analogous to the usual zeta function for a flow but associated to $\mathcal{C}(M, W)$, by

$$\zeta(s) = \prod_{\gamma \in \mathcal{C}'(M,W)} (1 - e^{-s|\gamma|})^{-1} = \exp \sum_{m=1}^{\infty} \sum_{\gamma \in \mathcal{C}'(M,W)} \frac{e^{-sm|\gamma|}}{m},$$

where $\mathcal{C}'(M, W)$ denotes the prime closed geodesics in $\mathcal{C}(M, W)$. This has abscissa of convergence $h = h(M)$. A similar function may be defined using the set $\mathcal{P}'$ of prime periodic T_ψ-orbits which intersect $\Sigma_A^+ \times \{e\}$:

$$Z(s) = \prod_{\tau \in \mathcal{P}'} (1 - e^{-s\lambda(\tau)})^{-1},$$

where, for $\tau = \{(x, g), T_\psi(x, g), \ldots, T_\psi^{n-1}(x, g)\}$, $\lambda(\tau) = r^n(x)$. (One can, of course, describe this in terms of a suspended semi-flow over $\Sigma_A^+ \times G$ but this would make the notation more cumbersome.) A standard calculation gives

$$Z(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{(x,g) \in \mathcal{P}_n} e^{-sr^n(x)},$$

where

$$\mathcal{P}_n = \{(x, g) : T_\psi^n(x, g) = (x, g) \text{ and } T_\psi^m(x, g) \in \Sigma_A^+ \times \{e\} \text{ for some } 0 \leq m < n\}.$$

It is this last function that will be related to Gurevič pressure.

The next lemma follows immediately from Lemma 2.5.2. In particular, the discrepancy between the number of periodic ϕ-orbits with $|\gamma| \leq T$ and the number of periodic T_ψ-orbits with $\lambda(\tau) = r^n(x) \leq T$ is at worst $O(e^{h'T})$.

Lemma 2.7.1. *$\zeta(s)/Z(s)$ is analytic and non-zero for $\mathrm{Re}(s) > h'$.*

Corollary 2.7.1. *$\zeta(s)$ has abscissa of convergence $h = h_0$ if and only if $Z(s)$ has abscissa of convergence h_0.*

Let

$$\mathcal{Q}_n = \{x : \sigma^n x = x \text{ and } \psi_n(x) = e\}.$$

Lemma 2.7.2. *For all $n \geq 1$, $\#\mathcal{Q}_n \leq \#\mathcal{P}_n \leq n\#\mathcal{Q}_n$. Hence, for $s \in \mathbb{R}$,*

$$\sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} e^{-sr^n(x)} \leq \sum_{\substack{T_\psi^n(x,g)=(x,g) \\ \exists 0 \leq m < n \,:\, T_\psi^m(x,g) \in \Sigma_A^+ \times \{e\}}} e^{-sr^n(x)} \leq n \sum_{\substack{\sigma^n x = x \\ \psi_n(x) = e}} e^{-sr^n(x)}.$$

Proof. Since $T_\psi^n(x, g) = (x, g)$ if and only if $\sigma^n x = x$ and $\psi_n(x) = e$, the first inequality follows by considering the injection $\mathcal{Q}_n \to \mathcal{P}_n : x \mapsto (x, e)$. On the other hand, when $T_\psi^n(x,g) = (x,g)$, the condition $T_\psi^m(x,g) \in \Sigma_A^+ \times \{e\}$ is equivalent to $\psi_m(x)g = e$, i.e. $g = \psi_m(x)^{-1}$. So, for each $\sigma^n x = x$ with $\psi_n(x) = 1$, $\{g \in G : (x,g) \in \mathcal{P}_n\} \subset \{1, \psi(x)^{-1}, \ldots, \psi_{n-1}(x)^{-1}\}$ and hence has cardinality at most n. This proves the second inequality. □

Now, as promised, we relate the abscissa of convergence of $Z(s)$ to Gurevič pressure.

Lemma 2.7.3. *The abscissa of convergence of $Z(s)$ is the unique real number ξ for which $P_{\mathrm{Gur}}(-\xi r, T_\psi) = 0$.*

Proof. It follows from Lemma 2.7.2 that

$$\sum_{x \in \mathcal{Q}_n} e^{-sr^n(x)} \leq \sum_{(x,g) \in \mathcal{P}_n} e^{-sr^n(x)} \leq n \sum_{x \in \mathcal{Q}_n} e^{-sr^n(x)}$$

and hence that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{(x,g)\in\mathcal{P}_n} e^{-sr^n(x)} = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in\mathcal{Q}_n} e^{-sr^n(x)}.$$

Since

$$P_{\mathrm{Gur}}(-\xi r, T_\psi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in\mathcal{Q}_n} e^{-\xi r^n(x)},$$

we have that $Z(\xi)$ converges if $P_{\mathrm{Gur}}(-\xi r, T_\psi) < 0$ and diverges if $P_{\mathrm{Gur}}(-\xi r, T_\psi) > 0$.

The Gurevič pressure is convex and hence continuous (Proposition 4.4 of [45]). Choose $\xi, \xi' \in \mathbb{R}$ with $\xi < \xi'$ Write $r_0 = \inf\{r^n(x)/n : \sigma^n x = x,\ n \geq 1\} > 0$. Then, for $\sigma^n x = x$, $e^{-\xi' r^n(x)} \leq e^{-\xi r^n(x)} e^{-n(\xi'-\xi)r_0}$. Thus

$$\begin{aligned} P_{\mathrm{Gur}}(-\xi' r, T_\psi) &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in\mathcal{Q}_n} e^{-\xi' r^n(x)} \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \left(e^{-n(\xi'-\xi)r_0} \sum_{x\in\mathcal{Q}_n} e^{-\xi r^n(x)} \right) \\ &= -(\xi'-\xi)r_0 + P_{\mathrm{Gur}}(-\xi r, T_\psi) < P_{\mathrm{Gur}}(-\xi r, T_\psi), \end{aligned}$$

so that $P_{\mathrm{Gur}}(-\xi r, T_\psi)$ is a strictly decreasing function. Furthermore, the transitivity of $T_\psi : \widetilde{\Sigma}^+ \to \widetilde{\Sigma}^+$ ensures that $P_{\mathrm{Gur}}(-\xi r, T_\psi)$ is not everywhere $-\infty$. Hence there is a unique $\xi \in \mathbb{R}$ such that $P_{\mathrm{Gur}}(-\xi r, T_\psi) = 0$. By the above characterisation, this is the abscissa of convergence of $Z(s)$. □

We may now prove our main result, formulated for closed geodesics.

Theorem 2.7.1. *Let Γ_0 be a convex cocompact group of isometries of a pinched Hadamard manifold X and let Γ be a normal subgroup of Γ_0. Then $h(X/\Gamma) = h(X/\Gamma_0)$ if and only if $G = \Gamma_0/\Gamma$ is amenable.*

Proof. By Lemma 2.5.3, we have $P(-h_0 r, \sigma) = 0$ and, by Proposition 2.3.1, $P_{\mathrm{Gur}}(-h_0 r, T_\psi) < P(-h_0 r, \sigma)$ unless G is amenable, in which case equality holds. Hence, if G is amenable then $P(-h_0 r, T_\psi) = 0$ and so $h = h(M) = h_0$. On the other hand, if G is not amenable then $P_{\mathrm{Gur}}(-\xi r, T_\psi) = 0$ for some $\xi < h_0$ and so, by Corollary 2.7.1 and Lemma 2.7.3, $h < h_0$. $\square$

Remark 2.7.1. We could also have proved that equality of critical exponents implies amenability directly by replacing Stadlbauer's result with a recent result of Jaerisch [22], in which the Gurevič pressure is replaced by the logarithm of the spectral radius of a transfer operator associated to T_ψ acting on a suitably chosen Banach space, together with some approximation arguments along the lines of those used in [37, 38].

Chapter 3

Critical exponents of normal subgroups, the spectrum of group extended transfer operators, and Kazhdan distance

For a pinched Hadamard manifold X and a discrete group of isometries Γ of X, the critical exponent δ_Γ is the exponential growth rate of the orbit of a point in X under the action of Γ. We show that the critical exponent for any family $\mathcal{N}$ of normal subgroups of Γ_0 has the same coarse behaviour as the Kazhdan distances for the right regular representations of the quotients Γ_0/Γ. The key tool is to analyse the spectrum of transfer operators associated to subshifts of finite type, for which we obtain a result of independent interest. The results of this chapter are due to the author and are contained in a pre-print [13].

3.1 Introduction

Let X be a simply connected, complete Riemannian manifold whose curvatures are bounded between two negative constants – this is sometimes called a *pinched Hadamard manifold*. For any non-elementary discrete group of isometries Γ of X, the Γ-orbit of a point inside a ball of radius R grows exponentially in R. More precisely, define the *critical exponent* δ_Γ (see also Chapter 2) by

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{1}{R} \log \# \{g \in \Gamma : d(x, gx) \leq R\}.$$

It is easy to see that the definition is independent of $x \in X$. When Γ is non-elementary, the limit exists and $\delta_\Gamma > 0$ – see, for instance, [35].

If Γ is torsion-free then we may form the quotient manifold $M = X/\Gamma$ and geodesic flow $\phi^t : SM \to SM$ on the unit tangent bundle SM. We refer to a closed geodesic γ in M and the corresponding periodic orbit γ in SM interchangeably. Write $\mathrm{Per}(\phi)$ for the collection of periodic orbits, and write $|\gamma|$ for the length of a geodesic and the period of the orbit. We say that a point $x \in SM$ is *wandering* if it is contained in a neighbourhood U such that $\phi^t U \cap U = \varnothing$ for all large t. The *non-wandering* set $\Omega(\phi)$ is the collection of points that are not wandering. Note that when M is compact, $\Omega(\phi) = SM$. Following [35], the *Gurevič pressure*, $\mathcal{P}(\phi)$, of the geodesic flow is defined by

$$\mathcal{P}(\phi) = \limsup_{T\to\infty} \frac{1}{T} \# \{\gamma \in \mathrm{Per}(\phi) : |\gamma| \leq T, \gamma \cap W \neq \varnothing\},$$

where W is any (non-empty) open subset of Ω with compact closure. (This was denoted by $h(M) = \mathcal{P}(\phi)$ in Chapter 2.) Note that [35] defines the Gurevič pressure $\mathcal{P}(\phi, F)$ more generally for a potential F, but in our case $F = 0$ and so we simplify the notation $\mathcal{P}(\phi, 0) = \mathcal{P}(\phi)$. Also contained in [35] is the proof that $\mathcal{P}(\phi) = \delta_\Gamma$ when $\mathcal{P}(\phi) > 0$ (which is satisfied in our later specialisation).

If Γ is cocompact, then

$$\delta_\Gamma = \lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}(x, R),$$

where $\mathrm{Vol}(x, R)$ is the volume of an R-ball around $x \in X$. For n-dimensional quaternionic hyperbolic space $\boldsymbol{H}^n_{\mathbb{H}}$, with $n \geq 2$, and the Cayley plane $\boldsymbol{H}^2_{\mathbb{O}}$, Corlette [11] showed that this value is isolated in the following way. If Γ is a lattice in $\mathrm{Isom}^+(\boldsymbol{H}^n_{\mathbb{H}})$, then $\delta_\Gamma = 4n + 2$; and otherwise $\delta_\Gamma \leq 4n$. There is also the corresponding statement for the Cayley plane: $\delta_\Gamma = 22$ when Γ is a lattice in $\mathrm{Isom}^+(\boldsymbol{H}^2_{\mathbb{O}})$; and otherwise $\delta_\Gamma \leq 16$. Therefore, in each case, for a fixed lattice Γ_0, there is a uniform gap $\delta_\Gamma < \delta_{\Gamma_0}$ for infinite index $\Gamma \leq \Gamma_0$. The symmetry of the spaces $\boldsymbol{H}^n_{\mathbb{H}}$ and $\boldsymbol{H}^2_{\mathbb{O}}$ are notable in the approach to this problem, which we discuss in the next section.

In this paper we develop a dynamical approach to analyse the critical exponent of normal subgroups $\Gamma \trianglelefteq \Gamma_0$, of a fixed (torsion-free) *convex cocompact* Γ_0. The *convex cocompact* hypothesis says that the geodesic flow $\phi_0^t : SM_0 \to SM_0$, where $M_0 = X/\Gamma_0$, has compact non-wandering set $\Omega(\phi_0)$.

For any $\Gamma \trianglelefteq \Gamma_0$, we have $\delta_\Gamma \leq \delta_{\Gamma_0}$ and moreover $\delta_\Gamma = \delta_{\Gamma_0}$ precisely when Γ_0/Γ is amenable (we discuss the history of this result later). Consequently $\delta_\Gamma < \delta_{\Gamma_0}$ when Γ_0/Γ is non-amenable. Our result is to describe coarse behaviour of

δ_Γ, over any family $\mathcal{N}$ of normal subgroups of Γ_0, in terms of *Kazhdan distances* associated to the quotients Γ_0/Γ, which we explain below. Some natural families of coverings are a tower of regular covers $M_1 \to M_2 \to \cdots \to M_0$, corresponding to a family $\Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_0$ of normal subgroups of Γ_0; and the family of all non-amenable regular covers of M_0, i.e. all $\Gamma \trianglelefteq \Gamma_0$ for which Γ_0/Γ is non-amenable.

In the following, G is assumed to be a countable group (however, many of the definitions can be made in the setting of locally compact groups). As in Chapter 2, we present the definition of an amenable group G due to Følner [16]. A group G is *amenable* if for every $\epsilon > 0$, and for every finite set A, there exists a set E which is $\epsilon, A-$invariant; that is,

$$\#E\Delta Ea \leq \epsilon \#E,$$

for all $a \in A$.

Write $\mathbb{1}_E \in \ell^2(G)$ for the indicator function on the set E. Noting that $\#E\Delta Ea = |\mathbb{1}_E - \mathbb{1}_{Ea}|$, there is the following equivalent definition in terms of the right regular representation $\pi_G : G \to \mathcal{U}(\ell^2(G))$, $(\pi_G(g)f)(x) = f(xg)$, due to Hulanicki [21]. A group G is *amenable* if and only if, for any finite generating set $A \subset G$,

$$\inf_{v\in\ell^2(G),|v|=1} \max_{a\in A} |\pi_G(a)v - v| = 0.$$

For any unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ in a Hilbert space $(\mathcal{H}, |\cdot|)$, the quantity $\kappa_A(\rho, \mathbb{1})$ defined by

$$\kappa_A(\rho, \mathbb{1}) := \inf_{v\in V,|v|=1} \max_{a\in A} |\rho(a)v - v|$$

is called the *Kazhdan distance (between ρ and the trivial representation $\mathbb{1}$)*. A group is said to have *property* (T) if there is some $\kappa > 0$ such that $\kappa_A(\rho, \mathbb{1}) > \kappa$ for all unitary representations that have no invariant vector.

The main theorem of this paper is the following.

Theorem 3.1.1. *Let Γ_0 be a convex cocompact group of isometries of a pinched Hadamard manifold X, and let A be a finite generating set for Γ_0. For any collection $\mathcal{N}$ of normal subgroups of Γ_0, we have*

$$\sup_{\Gamma\in\mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0} \text{ if and only if } \inf_{\Gamma\in\mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0.$$

We remark that this theorem is reminiscent of results on the bottom of the spectrum of the Laplacian (for example by Sunada [52]) which we discuss in the next section.

If Γ_0 has property (T), then we have that

$$\inf_{\Gamma \trianglelefteq \Gamma_0 : [\Gamma_0 : \Gamma] = \infty} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0.$$

We remark that, in this case, $[\Gamma_0 : \Gamma] = \infty$ is equivalent to Γ_0/Γ being non-amenable.

It is known that the isometry group of for real hyperbolic space $\boldsymbol{H}^n_{\mathbb{R}}$ does not have property (T), and so its cocompact subgroups also do not satisfy property (T). However, when we consider groups arising from variable curvature, we do find cocompact examples with property (T). Indeed, the mechanism behind the gap in the critical exponents for $\boldsymbol{H}^n_{\mathbb{H}}$ and $\boldsymbol{H}^2_{\mathbb{O}}$, as shown by Corlette, is the fact their isometry groups have property (T) [2].

Corollary 3.1.1. *With the hypotheses of Theorem 3.1.1, if* Γ_0 *has property (T) then*

$$\sup \{\delta_\Gamma : \Gamma \trianglelefteq \Gamma_0, [\Gamma_0 : \Gamma] = \infty\} < \delta_{\Gamma_0}.$$

The proof of Theorem 3.1.1 relies on an analysis of the dynamics of the geodesic flow, and in particular the symbolic dynamics for the geodesic flow. In this way, we relate the problem to the spectrum of group extended transfer operators. We prove an analogous theorem about the spectrum of group extended transfer operators which is of independent interest. This approach is a departure from the methods employed for the symmetric spaces.

3.2 Background and history

A classical example of the interplay between combinatorial properties of a group, and the geometry on which it acts, is given in Brooks [6], [7]. Let $M \to M_0$ be a regular covering of a Riemannian manifold M_0 of "finite topological type" (i.e. M_0 is the union of finitely many simplices). Let $\lambda_0(M)$ and $\lambda_0(M_0)$ denote the bottom of the spectrum of the Laplacian on M and M_0 respectively. Brooks shows that $\lambda_0(M) = \lambda_0(M_0)$ if and only if the group of deck transformation given by the covering is amenable. We will refer to a result of this form as *an amenability dichotomy.* This was extended by Sunada [52] in the following way. We assume now that M_0 is compact, and so $\lambda_0(M_0) = 0$, and write $M_0 = X/\Gamma_0$. For any $\Gamma \trianglelefteq \Gamma_0$ we get a regular cover $M_\Gamma = X/\Gamma$ of M_0. Sunada shows that for any finite generating set A of Γ_0, there are constants c_1, c_2 depending only on the geometry of X and on A, such that for any regular cover $M_\Gamma = X/\Gamma$, we have

$$c_1(\kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}))^2 \leq \lambda_0(M) \leq c_2(\kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}))^2,$$

where A/Γ denotes the projection of A to the quotient Γ_0/Γ. These results were also generalised by Roblin and Tapie [44], and in the thesis of Tapie [53], relating the difference $\lambda_0(M_0) - \lambda_0(M)$ to the bottom of the spectrum of a combinatorial Laplacian (which is in turn related to the Kazhdan distance).

A consequence of these spectral results is that, for any family of normal subgroups $\mathcal{N}$ of Γ_0, we have

$$\inf_{\Gamma \in \mathcal{N}} \lambda_0(X/\Gamma) = 0 \text{ if and only if } \inf_{\Gamma \in \mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) = 0.$$

For n-dimensional real hyperbolic space $\boldsymbol{H}^n_{\mathbb{R}}$, the spectral geometry and dynamics are related by a celebrated theorem due to Patterson [34] and Sullivan [51]. We have that,

$$\lambda_0(X/\Gamma) = \begin{cases} \delta_\Gamma(n-1-\delta_\Gamma) & \text{if } \delta_\Gamma \geq \frac{n-1}{2} \\ \frac{(n-1)^2}{4} & \text{if } \delta_\Gamma \leq \frac{n-1}{2}. \end{cases}$$

There are analogous statements for the other noncompact rank 1 symmetric spaces. However these results fail to extend to spaces which do not satisfy such strong symmetry hypotheses.

We now return to the setting of the introduction: X is a pinched Hadamard manifold and Γ_0 is a (torsion-free) convex cocompact group of isometries. In this context, various authors have developed more dynamical methods to obtain an analogue of the amenability dichotomy of Brooks. With the hypotheses we have given X and Γ_0, it was first showed by Roblin [43] that if Γ_0/Γ is amenable, then $\delta_{\Gamma_0} = \delta_\Gamma$; and recently Dougall and Sharp [14] have shown the converse, that $\delta_{\Gamma_0} = \delta_\Gamma$ implies that Γ_0/Γ is amenable. The difference in techniques is notable: Roblin's result was obtained by analysing Patterson-Sullivan measures on the boundary, whereas Dougall and Sharp exploit the symbolic dynamics for the geodesic flow – it is this latter approach that we extend. Another important reference (that is key to [14]) is that of Stadlbauer [50], who obtained the equivalence in the setting of $X = \boldsymbol{H}^n_{\mathbb{R}}$ and Γ_0 essentially free, and whose techniques we discuss later.

3.3 Transfer operators and group extensions

For a finite alphabet $\mathcal{W} = \{1, \ldots, k\}$, we can give rules governing when two letters in the alphabet can be concatenated in terms of a $k \times k$ matrix A with entries 0 or 1. Namely, for $i, j \in \mathcal{W}$, the concatenation ij is said to be *admissible* if $A(i,j) = 1$. In this way, the set of admissible words of length n, $\mathcal{W}^n$, is the collection of concatenations $w = x_0 \cdots x_{n-1}$, where $x_0, \dots, x_{n-1} \in \mathcal{W}$ and $A(x_i, x_{i+1}) = 1$ for all $i = 0, \cdots, n-2$. Extending this to one-sided infinite words, define the (one-sided) shift space to be

$$\Sigma^+ = \left\{ x_0 x_1 \cdots \in \mathcal{W}^{\mathbb{Z}^+} : \forall i \in \mathbb{Z}^+,\ A(x_i, x_{i+1}) = 1 \right\}.$$

The two-sided shift space Σ is defined analogously by

$$\Sigma = \left\{ \cdots x_{-1} x_0 x_1 \cdots \in \mathcal{W}^{\mathbb{Z}} : \forall i \in \mathbb{Z},\ A(x_i, x_{i+1}) = 1 \right\}.$$

For brevity, we make the following definitions for the two-sided space Σ. However, they pass to Σ^+ by the canonical projection $\Sigma \to \Sigma^+$, given by forgetting past (negative) coordinates.

Write x to denote an element of Σ, and write x^i for the sequence element at index i; in this way $x = (x^i)_{i \in \mathbb{Z}}$. Similarly, for an element $w \in \mathcal{W}^n$, we write w^i to denote the ith element in the concatenation. There is a natural dynamical system, $\sigma : \Sigma \to \Sigma$ called the *shift map*, with defining property $\sigma(x)^i = x^{i+1}$. Together, we call the pair (Σ, σ) a *subshift of finite type*, or in some literature, a *topological Markov chain*.

There is a natural topology with basis consisting of cylinder sets

$$[w]_j^k := \left\{ x \in \Sigma : \forall j \leq i \leq k,\ x^i = w^{i-j} \right\},$$

for any $j, k \in \mathbb{N}$, and $w \in \mathcal{W}^{k-j+1}$. For the one-sided shift space Σ^+, we often write $[w] = [w]_0^{n-1}$, where $w \in \mathcal{W}^n$. The topology is metrizable: for any $0 < \theta < 1$ define the metric d_θ by

$$d_\theta(x, y) = \theta^{\inf\left\{ |i| \,:\, x^i \neq y^i \right\}}$$

and $d_\theta(x, x) = 0$.

We always assume that A is *aperiodic*, that is to say there is some $N > 0$ for which $A^N(i,j) > 0$ for all $i, j \in \{1, \dots k\}$. We call the minimal N the *aperiodicity constant*. The assumption that A is aperiodic is equivalent to σ being *topologically mixing*, i.e. for any non-empty open sets $U, V \subset \Sigma$ there is N for which $\sigma^{-n}(U) \cap V \neq \varnothing$ for all $n \geq N$.

The *Hölder continuous functions* $f : \Sigma \to \mathbb{R}$ are defined by the existence of $0 < \alpha \leq 1$ (a *Hölder exponent*) and C such that for any $x, y \in \Sigma$,

$$|f(x) - f(y)| \leq C d_\theta(x,y)^\alpha.$$

Notice that replacing θ by $\theta^{1/\alpha}$ gives that f has Hölder exponent 1 in this new metric. We will later fix a Hölder continuous r, and then assume that the metric is chosen to give r Hölder exponent 1.

For a strictly positive Hölder continuous function $r : \Sigma \to \mathbb{R}$, define the *suspension space* Σ_r (with *suspension* r) by

$$\Sigma_r = \Sigma \times \mathbb{R} / \sim_r,$$

where $\sim_r$ is the equivalence relation $(x, s) \sim_r (\sigma x, s - r(x))$. We define the suspension flow $\sigma_r^t : \Sigma_r \to \Sigma_r$, locally by $\sigma_r^t(x, s) = (x, s + t)$. The *pressure* $P(f, \sigma)$ of a Hölder continuous $f : \Sigma \to \mathbb{R}$ is defined to be

$$P(f, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{x \in \Sigma: \\ \sigma^n x = x}} e^{f^n(x)}.$$

We now specialise to the one-sided shift space Σ^+. For a Hölder continuous $r : \Sigma^+ \to \mathbb{R}$, we define the *transfer operator* $L_r : C(\Sigma^+, \mathbb{R}) \to C(\Sigma^+, \mathbb{R})$ by

$$L_r f(x) = \sum_{\substack{y \in \Sigma^+: \\ \sigma y = x}} e^{r(y)} f(y),$$

where $C(\Sigma^+, \mathbb{R})$ is the Banach space of continuous functions with the supremum norm $\|\cdot\|_\infty$. We write $\mathrm{spr}(L_r)$ for the spectral radius of L_r in this Banach space, omitting explicit reference to the space. We find better spectral properties for L_r when we restrict to the smaller Banach space of Hölder continuous functions. Write F_θ^+ for the functions $f : \Sigma^+ \to \mathbb{R}$ that are Lipschitz (have Hölder exponent 1) in the d_θ metric. Define the semi-norm

$$|f|_\theta = \sup_{n \in \mathbb{N}} \sup_{\substack{x,y: \\ x^i = y^i,\, |i| \leq n}} \frac{|f(x) - f(y)|}{\theta^n}.$$

Then $(F_\theta, \|\cdot\|_\theta)$ is a Banach space with the norm $\|\cdot\|_\theta = \|\cdot\|_\infty + |\cdot|_\theta$. By the Ruelle-Perron-Frobenius theorem [33], L_r has a simple, isolated, maximal eigenvalue at $e^{P(r,\sigma)}$.

For a countable group G, define the *group extension (with skewing function $\psi : \Sigma^+ \to G$)*, $T = T_\psi : \Sigma^+ \times G \to \Sigma^+ \times G$, to be the product space $\Sigma^+ \times G$

together with dynamical system

$$T(x, g) = (\sigma x, g(\psi(x))^{-1})$$

(Note that in the paper [14] the group extension by ψ is defined to be $T(x, g) = (x, g\psi(x))$, and so to translate to the present terminology we need to take the inverse ψ^{-1}. We have chosen this convention so as to simplify the expression for the transfer operator.) In the following ψ is always assumed to depend only on one letter. In this way, we can think of ψ as a function $\psi : \mathcal{W} \to G$. Moreover, for every $n \in \mathbb{N}$, we define $\psi^n : \Sigma^+ \to G$, $\psi^n(x) = \psi(x^0) \cdots \psi(x^{n-1})$, and write $\psi^n : \mathcal{W}^n \to G$, $\psi^n(w) = \psi(w^0) \cdots \psi(w^{n-1})$.

For $r : \Sigma^+ \to \mathbb{R}$, there is a unique $\widetilde{r} : \Sigma^+ \times G \to \mathbb{R}$ such that $\widetilde{r}(x, g) = r(x)$. We therefore dispense with the cumbersome tilde, and simply write this function as $r : \Sigma^+ \times G \to \mathbb{R}$. Define the *group extended transfer operator* $\mathcal{L}_r$ pointwise by

$$\mathcal{L}_r f(x, g) = \sum_{\substack{(y,g^*) \in \Sigma^+ \times G: \\ T(y,g^*)=(x,g)}} e^{r(y)} f(y, g^*) = \sum_{\substack{y \in \Sigma^+: \\ \sigma y = x}} e^{r(y)} f(y, g\psi(y)),$$

and so

$$\mathcal{L}_r^n f(x, g) = \sum_{\substack{(y,g^*) \in \Sigma^+ \times G: \\ T^n(y,g^*)=(x,g)}} e^{r^n(y)} f(y, g^*) = \sum_{\substack{y \in \Sigma^+: \\ \sigma^n y = x}} e^{r(y)} f(y, (g\psi(\sigma^{n-1} y) \cdots \psi(y)).$$

Define the Banach space $(\mathcal{C}^\infty, \| \cdot \|)$ by

$$\mathcal{C}^\infty = \left\{ f \in C(\Sigma^+ \times G, \mathbb{R}) : \|f\| < \infty \right\},$$

$$\|f\| = \sqrt{\sum_{g \in G} \sup_{x \in \Sigma^+} |f(x, g)|^2}.$$

Then $\mathcal{L}_r : \mathcal{C}^\infty \to \mathcal{C}^\infty$ is a bounded operator. We will always take the spectral radius $\mathrm{spr}(\mathcal{L}_r)$ with respect to the $\mathcal{C}^\infty$ norm, and continue to omit reference to the space.

Following Sarig, [46], define the *Gurevič pressure* $P_{\mathrm{Gur}}(f, T_\psi)$ of a Hölder continuous $f : \Sigma^+ \times G \to \mathbb{R}$ by

$$P_{\mathrm{Gur}}(f, T_\psi) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\substack{\sigma^n x = x: \\ \psi^n(x) = e_G}} e^{f^n(x)},$$

where e_G is the identity of G.

In general we have that if T_ψ is topologically transitive then $P_{\text{Gur}}(r, T_\psi) \leq \log \operatorname{spr}(\mathcal{L}_r) \leq \log \operatorname{spr}(L_r) = P(r, \sigma)$. If we assume in addition that the pair (ψ, r) is *weakly symmetric*, in the following sense, then we have that $P_{\text{Gur}}(r, T_\psi) = \log \operatorname{spr}(\mathcal{L}_r)$ [22]. We say an involution $\dagger$ on $\mathcal{W}$ is *weakly symmetric with respect to* r if there are real numbers D_n such that $D_n^{1/n} \to 1$ and

$$\sup_{x \in [w], y \in [w^\dagger]} \frac{e^{r^n(x)}}{e^{r^n(y)}} \leq D_n$$

for every $w \in \mathcal{W}^n$, where $w^\dagger := (w^{n-1})^\dagger \cdots (w^0)^\dagger$. We then say that the pair (ψ, r) is *weakly symmetric* if there is an involution $\dagger$ such that $\psi(v^\dagger) = \psi(v)^{-1}$ for all $v \in \mathcal{W}$, and such that $\dagger$ is weakly symmetric with respect to r.

It will be useful to extend the definition of weak symmetry to Σ. We say that $\dagger$ is symmetric with respect to $r : \Sigma \to \mathbb{R}$ if there is some $\beta(n)$ with $\beta(n)/n \to 0$ as $n \to \infty$, such that $|r^n(x) - r^n(x^\dagger)| \leq \beta(n)$ for any $x \in \Sigma$ with $\sigma^n x = x$, and $x^\dagger \in \Sigma$ defined by $(x^\dagger)^i = (x^{-i})^\dagger$. In section 3.6 we show that in the one-sided case Σ^+, the two definitions are equivalent (once we interperet $x^\dagger \in \Sigma^+$ as $(x^\dagger)^i = (x^{kn-i})^\dagger$, for $i = 0, \ldots, n-1$ and $k \in \mathbb{N}$, for a point x of period n).

Proposition 3.3.1 (Stadlbauer [50])**.** *Assume $T_\psi : \Sigma^+ \times G \to \Sigma^+ \times G$ is transitive. If G is non-amenable, then* $\operatorname{spr}(\mathcal{L}_r) < \operatorname{spr}(L_r)$. *Assuming that (ψ, r) is weakly symmetric, the converse holds: if G is amenable, then $P_{\text{Gur}}(r, T_\psi) = P(r, \sigma)$*

Remark 3.3.1. The statement of Stadlbauer's theorem is actually for the Banach space

$$\mathcal{H}^\infty = \left\{ f : \Sigma^+ \times G \to \mathbb{R} : f(\cdot, g) \in L^1(\Sigma^+, \mu_r) \text{ for all } g \in G, \|f\|_{\mathcal{H}^\infty} < \infty \right\},$$

$$\|f\|_{\mathcal{H}^\infty} = \sqrt{\sum_{g \in G} \left(\int |f(x, g)| d\mu_r \right)^2}.$$

where μ_r is the equilibrium state for r. However, the proof only uses the fact that the spectrum is attained on the subset

$$\left\{ f \in \mathcal{H}^\infty : f(x, g) = f(y, g) \text{ for all } x, y \in \Sigma^+, g \in G \right\}$$

which is isometrically isomorphic to $\ell^2(G)$, and so the statement is true for any Banach space with this property (we will show that $\mathcal{C}^\infty$ has this property later). Moreover, Stadlbauer considers countable alphabets under a certain finiteness condition, namely having *big images and pre-images*. As the application we have in mind is for a finite alphabet we limit ourselves to the finite alphabet case.

Remark 3.3.2. Jaerisch [22] shows that we have the conclusion $\mathrm{spr}(\mathcal{L}_r) = \mathrm{spr}(L_r)$ if G is amenable, under no symmetry hypothesis.

In this paper we study a family of group extensions which can be seen as quotients of a fixed group extension $\psi : \Sigma^+ \to G$. For each $H \trianglelefteq G$, write $\psi_H(x)$ for the coset of G/H given by $\psi(x)$. In this way $T_{\psi_H} : \Sigma^+ \times G/H \to \Sigma^+ \times G/H$ is a group extension with skewing function ψ_H. For notational convenience, write $\mathcal{L}_{r,H}$ for the transfer operator given by r and T_{ψ_H}; and write $\mathcal{C}_H^\infty$ for the Banach space assosciated to $\mathcal{L}_{r,H}$. We also consider a family of transfer operators $\mathcal{L}_{r_s,H}$ where $s \mapsto r_s \in F_\theta$, $s \in [-1,1]$, is continuous in the $\|\cdot\|_\theta$ topology.

By Proposition 3.3.1, if $H \trianglelefteq G$ with G/H non-amenable and $T_{\psi_H} : \Sigma^+ \times G/H \to \Sigma^+ \times G/H$ is transitive, then $\mathrm{spr}(\mathcal{L}_{r,H}) < \mathrm{spr}(L_r)$. The proof in [50] finds an upper bound for $\mathrm{spr}(\mathcal{L}_{r,H})$ that depends on the first return to a cylinder under T_{ψ_H}. As this bound does not suffice for our needs, we introduce a new condition on ψ that removes this dependency.

Definition 3.3.1. We say that (Σ^+, G, ψ) satisfies *linear visibility with remainder* (LVR) if there exists a map $\chi : G \to \bigcup_{n=1}^\infty \mathcal{W}^n$ with the following properties:

- (visibility with remainder) there exists a finite set $\mathcal{R} \subset G$ such that for every $g \in G$, there are $r_1, r_2 \in \mathcal{R}$ with $\psi^{k_g}(\chi(g)) = r_1 g r_2$, where k_g is the length of the word $\chi(g)$;
- (linear growth) there exists L such that for any finite collection $g_1, \ldots, g_r \in G$, writing $g = g_1 \cdots g_r$, we have that $k_g \leq L(\sum_{i=1}^r k_{g_i})$, where k_g is the length of $\chi(g)$, and k_{g_i} the length of $\chi(g_i)$, for each i.

3.4 Summary of Main Results

We restate our theorem about the behaviour of the critical exponent.

Theorem 3.1.1. *Let Γ_0 be a convex cocompact group of isometries of a pinched Hadamard manifold X, and let A be a finite generating set for Γ_0. For any collection $\mathcal{N}$ of normal subgroups of Γ_0, we have*

$$\sup_{\Gamma \in \mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0} \text{ if and only if } \inf_{\Gamma \in \mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0.$$

As in [14], the proof of the theorem uses the dynamics of the geodesic flow $\phi_\Gamma^t : SM_\Gamma \to SM_\Gamma$, for $M_\Gamma = X/\Gamma$, and in turn the dynamics of group extended shift spaces. Therefore it will be crucial to prove the following theorem about the spectrum of group extended transfer operators. This extends the results of Stadlbauer [50].

As in the introduction, fix $\sigma : \Sigma^+ \to \Sigma^+$ a topologically mixing subshift of finite type, and $r : \Sigma^+ \to \mathbb{R}$ a potential. Fix G a countable group and $\psi : \Sigma^+ \to G$ constant on cylinders of length 1.

Theorem 3.4.1. *Let A be a finite generating set for G, and let $\mathcal{N}$ be a collection of normal subgroups of G.*

(i) Assume that (ψ, r) is weakly symmetric. Then

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) = 0 \implies \sup_{H \in \mathcal{N}} P_{\text{Gur}}(r, T_{\psi_H}) = P(r, \sigma).$$

(ii) Assume that (Σ^+, G, ψ) satisfies (LVR). *Then*

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) > 0 \implies \sup_{H \in \mathcal{N}} \operatorname{spr}(\mathcal{L}_{r,H}) < \operatorname{spr}(L_r).$$

(iii) In addition, in case (ii) suppose that $s \mapsto r_s$ is continuous (in the $\| \cdot \|_\theta$ topology) for $s \in [-1, 1]$. Then

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) > 0 \implies \sup_{H \in \mathcal{N}, s \in [-\delta, \delta]} \operatorname{spr}(\mathcal{L}_{r_s,H}) < \operatorname{spr}(L_{r_0}),$$

for some $\delta > 0$.

Remark 3.4.1. If $\psi : \Sigma^+ \to G$ depends on n-coordinates, as opposed to one, then we may still apply the conclusions of Theorem 3.4.1. To see this, let Σ_n^+ denote the subshift of finite type whose alphabet is given by admissible words of length n for Σ^+, and with transition matrix $A_n(u, v) = 1$ if and only if $u^{i+1} = v^i$ for all $i = 0, \ldots, n-2$. Then ψ gives rise to $\psi_n : \Sigma_n^+ \to G$ depending only on one coordinate. Moreover, the statistics for the Hölder continuous functions, pressure and transfer operators pass to (Σ_N^+, ψ_n) in the natural way.

3.5 Axiom A flows and symbolic dynamics

We now take a brief excursion into the theory of Smale's Axiom A flows [49] and the symbolic coding of Bowen [5]. We refresh the material of Chapter 2, Subsection 2.5 for the reader's convenience.

Throughout, f^t is a smooth flow on a complete Riemannian manifold N.

A closed, f^t-invariant set $\Lambda \subset N$ is said to be *hyperbolic* if there is a continuous, Df^t-invariant splitting of the tangent bundle

$$T_\Lambda(N) = E^0 \oplus E^s \oplus E^u$$

and constants $\lambda, C > 0$ such that

- E^0 the line bundle tangent to the flow direction;
- $\|Df^t v\| \leq Ce^{-\lambda t}\|v\|$ for all $v \in E^s$;
- $\|Df^{-t} v\| \leq Ce^{-\lambda t}\|v\|$ for all $v \in E^u$;

We remark that this definition is independent of the choice of metric when Λ is compact.

The set Λ is said to be a *basic set* if

1. Λ is compact and hyperbolic;
2. f^t_Λ is transitive;
3. the periodic orbits for f^t_Λ are dense in Λ; and
4. there is an open neighbourhood U of Λ such that $\bigcap_{t\in\mathbb{R}} f^t(U) = \Lambda$.

The flow f^t satisfies Smale's *Axiom A* if the non-wandering set $\Omega(f)$ is a finite union of basic sets.

Let $\phi_0^t : SM_0 \to SM_0$ be the geodesic flow given by $M_0 = X/\Gamma_0$ where Γ_0 is convex cocompact. Then the non-wandering set $\Omega(\phi_0)$ is a basic set. (See for instance [24, Chapter 17] in the case that M is compact.)

We now describe some of the constructions relating to the theory of (hyperbolic) basic sets, which play an important role in Bowen's symbolic coding.

For $x \in \Omega(f)$ define the (strong) *local stable manifold* $W^s_\epsilon(x)$ and (strong) *local unstable manifold* $W^u_\epsilon(x)$ by

$$W^s_\epsilon(x) = \left\{ y \in N : d(f^t(x), f^t(y)) \leq \epsilon \text{ for all } t, \lim_{t\to\infty} d(f^t(x), f^t(y)) = 0 \right\}$$

$$W^u_\epsilon(x) = \left\{ y \in N : d(f^{-t}(x), f^{-t}(y)) \leq \epsilon \text{ for all } t, \lim_{t\to\infty} d(f^{-t}(x), f^{-t}(y)) = 0 \right\}$$

For small enough ϵ, these sets are diffeomorphic to embedded disks of codimension 1. These sets give us a *local product structure* $[\cdot,\cdot]$. For sufficiently close x, y, we have that $W^s_\epsilon(x) \cap W_\epsilon(\phi_0^t(y)) \neq \varnothing$ for a unique $t \in [-\epsilon, \epsilon]$, and we define $[x, y]$ to be this intersection point.

Suppose that $D^1, \ldots, D^k$ are codimension 1 disks that form a local cross-section to the flow. We say that $S^i \subset \mathrm{int}(D^i) \cap \Omega(f)$ is a *rectangle* if $x, y \in S^i$ implies that $[x,y] = f^t z$, for some $z \in D^i$, $t \in [-\epsilon, \epsilon]$. We say that S^i is *proper* if $\overline{\mathrm{int}(S^i)} = S^i$, where the interior is taken relative to $D^i \cap \Omega(f)$.

Write P for the Poincaré map $P : \bigcup_{i=1}^k S^i \to \bigcup_{i=1}^k S^i$. Write $W_\epsilon^s(x, S^i)$ and $W_\epsilon^u(x, S^i)$ for the projection of $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ onto $\mathrm{int}(S^i)$ respectively. We say that $\{S^1, \cdots, S^k\}$ is a *Markov section* if

- $x \in \mathrm{int}(S^i)$ and $Px \in \mathrm{int}(S^j)$ implies $P(W_\epsilon^s(x, S^i)) \subset W_\epsilon^s(Px, S^j))$; and
- $x \in \mathrm{int}(S^i)$ and $P^{-1}x \in \mathrm{int}(S^j)$ implies $P^{-1}(W_\epsilon^u(x, S^i)) \subset W_\epsilon^s(P^{-1}x, S^j)$.

Proposition 3.5.1 (Bowen [5]). *For all sufficiently small $\epsilon > 0$, f^t has a Markov section $\{S^1, \cdots, S^k\}$ such that* $\mathrm{diam}(S^i) \leq \epsilon$ *for each i, and*

$$\bigcup_{t \in [-\epsilon, \epsilon]} f^t(\cup_{i=1}^k S^i) = \Omega(f).$$

These Markov sections provide us with a 'symbolic coding' for the geodesic flow. In the following, the Markov section $\{S^1, \cdots, S^k\}$ plays the role of an alphabet for a subshift of finite type Σ with transition matrix A, defined by $A(i,j) = 1$ if there is $x \in \mathrm{int}(S^i)$ with $Px \in \mathrm{int}(S^j)$.

We specialise to the geodesic flow ϕ_0^t for the statement of the concluding proposition. We write $N_{\phi_0^t}(T)$ and $N_{\sigma_r^t}(T)$ for the number of periodic orbits of ϕ_0^t and σ_r^t, respectively, whose period is at most T. It is well known that $\delta_{\Gamma_0} = \lim_{T \to \infty} \frac{1}{T} \log N_{\phi_0^t}(T)$.

Proposition 3.5.2 (Bowen [5]). *There is a mixing subshift of finite type $\sigma : \Sigma \to \Sigma$, a positive Hölder potential $r : \Sigma \to \mathbb{R}^+$ such that the suspended flow $\sigma_r^t : \Sigma^r \to \Sigma^r$ is semi-conjugate to $\phi_0^t : \Omega(\phi_0) \to \Omega(\phi_0)$, i.e. there is a Hölder continuous $\theta : \Sigma^r \to \Omega(\phi_0)$ such that $\theta \circ \sigma_r^t = \phi_0^t \circ \theta$. Although θ is not a bijection, we have $N_{\sigma_r^t}(T) = N_{\phi_0^t}(T) + O(e^{h'T})$, for some $h' < \delta_{\Gamma_0}$.*

We write $\{S_0^1, \ldots, S_0^k\}$ for the Markov section for ϕ_0^t. By Adachi [1] and Rees [42] we may assume the sections have been chosen to reflect the involution $\iota : SM_0 \to SM_0$, $\iota(x,v) = (x, -v)$. That is, we may assume that there is a fixed point free involution $\dagger$ on $\{1, \cdots, k\}$ such that $\iota(S_0^i) = S_0^{i^\dagger}$ for each i.

3.6 Proof of Theorem 3.1.1

Write $p : SX \to SM_0$ for the induced covering map between tangent spaces, and $\pi : SX \to X$ for the usual projection. Fix $S^i \subset SX$ such that $p^{-1} : S_0^i \subset$

$SM_0 \to S^i \subset SX$ is an isometry; and assume that these have been chosen with $\pi(S^i) = \pi(S^{i^\dagger})$. Write $\mathcal{S} = \bigcup_{g \in \Gamma_0} \bigcup_{i=1}^k gS^i$, and P for the Poincaré map $P : \mathcal{S} \to \mathcal{S}$. Define $\psi : \Sigma^+ \to \Gamma_0$ by $\psi(x_0x_1) = g$ if there is $z \in \mathrm{int}(S^{x_0})$ such that $Pz \in \mathrm{int}(gS^{x_1})$. We will verify that ψ is well-defined in the following proposition. For $\Gamma \trianglelefteq \Gamma_0$, recall that we write ψ_Γ for projection of ψ to Γ_0/Γ. Write $s = h_\Gamma$ for the (unique) value for which $P_{\mathrm{Gur}}(-sr, T_{\psi_\Gamma}) = 0$.

Proposition 3.6.1 (Dougall-Sharp [14])**.** *The map ψ is well-defined, and (ψ, r) is weakly symmetric. When Γ is non-trivial, T_{ψ_Γ} is transitive. If $h_\Gamma > h'$ then $h_\Gamma = \delta_\Gamma$; otherwise $\delta_\Gamma \le h'$, where h' is the constant from Proposition 3.5.2.*

We give an indication of the proof.

Proof. For each i, j with $A(i,j) = 1$, fix a simply connected ball V_0 containing the pair S_0^i, S_0^j (we may assume that ϵ was chosen sufficiently small to allow this). There is a unique connected component V of $p^{-1}(V_0)$ containing S^i. Let $g \in \Gamma_0$ be the unique element for which $gS^j \subset V$. It follows that $\psi(ij) = g$; and therefore that ψ is well-defined.

We show that $\dagger$ is weakly symmetric with respect to r. Let $x \in \Sigma$ be a point of period n, and write $x^\dagger \in \Sigma$ for the sequence defined by $(x^\dagger)^{-i} = (x^i)^\dagger$. Then x and $x^\dagger$ determine periodic points $z = \theta(x), z^\dagger = \theta(x^\dagger) \in SM_0$, which are related by $\iota(z) = z^\dagger$. Therefore $z, z^\dagger$ have identical period $T = r^n(x) = r^n(x^\dagger)$, and so $|r^n(x) - r^n(x^\dagger)| = 0$ as required. Now, to complete the proof that (ψ, r) is weakly symmetric, we observe that the symmetry of the rectangles $\iota(S_0^i) = S_0^{i^\dagger}$ and the fact that we chose $S^i, S^{i^\dagger}$ to satisfy $\pi(S^i) = \pi(S^{i^\dagger})$ implies that $\psi(ij) = \psi(j^\dagger i^\dagger)^{-1}$.

Fix $\Gamma \trianglelefteq \Gamma_0$. Recall that we write $M_\Gamma = X/\Gamma$ and $\phi_\Gamma^t : SM_\Gamma \to SM_\Gamma$ for the geodesic flow. We give an equivalent definition of ψ_Γ. Write $S_\Gamma^i \subset SM_\Gamma$ for the projection of S^i to SM_Γ. Write $\mathcal{S}_\Gamma = \bigcup_{g \in \Gamma_0/\Gamma} \bigcup_{i=1}^k gS_\Gamma^i$, and P_Γ for the Poincaré map $P_\Gamma : \mathcal{S}_\Gamma \to \mathcal{S}_\Gamma$. Then we have that $\psi_\Gamma(ij) = g \in \Gamma_0/\Gamma$ if and only if there is $z \in \mathrm{int}(S_\Gamma^i)$ such that $P_\Gamma z \in \mathrm{int}(gS_\Gamma^j)$. The transitivity of T_{ψ_Γ} for any non-trivial Γ is therefore inherited by the transitivity of the geodesic flow.

Let $N_{\phi_0^t}(T, \Gamma)$ denote the number of orbits of ϕ_0^t whose period is at most T and whose lift to SM_Γ is closed. Analogously, let $N_{\sigma_r^t}(T, \Gamma)$ denote the periodic orbits of σ_r^t whose period is at most T and whose lift in the r-suspension over $T_{\psi_\Gamma} : \Sigma \times \Gamma_0/\Gamma \to \Sigma \times \Gamma_0/\Gamma$ is closed. Let γ be a periodic ϕ_0^t-orbit that passes through only the interior of rectangles. Then γ has a closed lift in SM_Γ, if and only if its pre-image under the semi-conjugacy has a closed lift in the r-suspension over $T_{\psi_\Gamma} : \Sigma \times \Gamma_0/\Gamma \to \Sigma \times \Gamma_0/\Gamma$. (We use the equivalent definition of ψ_Γ to observe this.)

In our setting, the "lim sup" in the definition of the Gurevič pressure is in fact a limit **??**. Moreover, $\mathcal{P}(\phi_\Gamma) = \lim_{T\to\infty} \frac{1}{T} \log N_{\phi_0^t}(T, \Gamma)$. To see this, note that for $\mathrm{diam}(W) = \epsilon$, $g\gamma \cap W \neq \varnothing$ for at most $|\gamma|/\epsilon$ different $g \in \Gamma_0/\Gamma$. Now, note that $h_\Gamma = \limsup_{T\to\infty} \frac{1}{T} \log N_{\sigma_r^t}(T, \Gamma)$. The following inequalities are given by our equivalent definition of ψ_Γ, and the semi-conjugacy in Proposition 3.5.2,

$$N_{\sigma_r^t}(T, \Gamma) - O(e^{h'T}) \leq N_{\phi_0^t}(T, \Gamma) \leq N_{\sigma_r^t}(T, \Gamma).$$

Therefore, if $h_\Gamma > h'$ then we must have that $h_\Gamma = \delta_\Gamma$. Since $\delta_\Gamma \leq h_\Gamma$ we deduce that $h_\Gamma \leq h'$ implies that $\delta_\Gamma \leq h'$ too. □

We will use of the following constructions relating to the visual boundary ∂X of X. We can give the unit tangent bundle SX Hopf coordinates (x, y, t), with respect to some fixed $o \in X$. (We suppress the dependency of the base-point o as we only make statements up to reparamaterising the geodesic paths.) For any $x, y \in X$, denote by $[x, y]$ the geodesic segment from x to y. If $x, y \in \partial X$, write $[x, y]$ for the geodesic path with Hopf coordinates $(x, y, t)_{t\in\mathbb{R}}$ (up to reparameterisation). Let $L(\Gamma_0)$ be the limit set of Γ_0. Write $\Omega = p^{-1}(\Omega(\phi_0))$. We have that Ω is equivalently characterised as the set of vectors that whose Hopf coordinates are (x, y, t), with $x, y \in L(\Gamma_0)$. In order to verify the (LVR) condition from Definition 3.6.1 we use the following claim, whose proof is deferred until later.

Claim 3.6.1. *There is a constant $R > 0$ such that for any $g \in \Gamma_0$, there is a geodesic γ in Ω passing within distance R of x and of gx, and moreover γ does not intersect the boundary of any rectangle.*

Write $\mathcal{I}$ for the collection of ϕ-orbit segments whose initial and terminal points lie in $\mathcal{S}$, and such that they do not intersect the boundary of any rectangle gS^i. We construct a map $\tau : \mathcal{I} \to \bigcup_n \mathcal{W}^n$. Let $\gamma \in \mathcal{I}$, and write its initial point as z. There is n such that γ is partitioned into orbit segments between $P^i z$ and $P^{i+1} z$, $i = 0, \ldots, n-1$. Define $\tau(\gamma) = w \in \mathcal{W}^n$ by $P^i z \in g_i S^{w^i}$, for some (unique) g_i, for each $i = 0, \ldots, n-1$.

Lemma 3.6.1. (Σ, ψ, r) *satisfies* (LVR).

Proof. We construct a map $\lambda : \Gamma_0 \to \mathcal{I}$, and define $\chi : \Gamma_0 \to \bigcup_n \mathcal{W}^n$ by $\chi = \tau \circ \lambda$. Fix some $x \in X$. For each $T > 0$, write $\mathcal{R}(T) \subset \Gamma_0$ for the elements $h \in \Gamma_0$ such that $\pi(hS^i)$ has distance at most T to x, for some i. Notice that this set is necessarily finite.

Let $g \in \Gamma_0$ be arbitrary. Let γ be the geodesic given in Claim 3.6.1. Let $y^1, y^2 \in SX$ be tangent to γ with $d(\pi(y^1), x), d(\pi(y^2), gx) \leq R$. Let $z^1, z^2 \in$

$\mathcal{S}$ with $\phi^s z^1 = y^1$ and $\phi^t z^2 = y^2$ for some $0 \leq s, t \leq \epsilon$. It follows that $d_X(\pi(z^1), x), d_X(\pi(z^2), gx) \leq R + 2\epsilon$. Define $\lambda(g) = [\pi(z^1), \pi(z^2)]$. We write w_g for $\chi(g) = \tau(\lambda(g))$ and k_g for the length of w_g. There are (unique) $h_1, h_2 \in \mathcal{R}(R + 2\epsilon)$ and j_1, j_2, such that $z^1 \in h_1 S^{j_1}$ and $z^2 \in g h_2 S^{j_2}$. Moreover, from the definition of ψ, we have that $g h_2 = h_1 \psi^{k_g - 1}(w_g)$. Therefore $\psi^{k_g-1}(w_g) = h_1^{-1} g h_2$, verifying the first part of (LVR).

It remains to show the 'linear' part of (LVR). First, note that the length $|\lambda(g)|$ of the orbit segement $\lambda(g)$ satisfies

$$d(x, gx) - 2R - 2\epsilon \leq |\lambda(g)| \leq d(x, gx) + 2R + 2\epsilon.$$

By the semi-conjugacy with ϕ_0^t, we have that $r^{k_g - 1}(w_g z_g) = |\lambda(g)|$ for some $z_g \in \Sigma$.

Let $g_1, \cdots, g_m \in \Gamma_0$ be arbitrary. Write $h_i = g_1 \cdots g_i$, $h_0 = e$, for each $i = 1, \ldots, m$. We have

$$|\lambda(h_m)| \leq 2R + 2\epsilon + \sum_{i=1}^m d(h_{i-1}x, h_i x) \leq (m+1)(2R + 2\epsilon) + \sum_{i=1}^m |\lambda(g_i)|,$$

and so

$$\begin{aligned}\min(r)(k_{h_m} - 1) &\leq (m+1)(2R + 2\epsilon) + \max(r) \sum_{i=1}^m k_{g_i} \\ &\leq (\max(r) + 2R + 2\epsilon + 1) \sum_{i=1}^m k_{g_i},\end{aligned}$$

as required. $\square$

We now prove the claim.

Claim 3.6.1. *There is a constant $R > 0$ such that for any $g \in \Gamma_0$, there is a geodesic γ in Ω passing within distance R of x and of gx, and moreover γ does not intersect the boundary of any rectangle.*

Proof. We make use of the following material from the Patterson-Sullivan theory for X and Γ_0. Our account is based on [35]. Since $\delta_{\Gamma_0} < \infty$, there exists a Patterson-Sullivan family $\{\mu_x\}_{x \in X}$ whose support is precisely the limit set $L(\Gamma_0)$ (and has dimension δ_{Γ_0}). For a subset $A \subset X$, and a point $x \in X \cup \partial X$, define $\mathcal{O}_x A \subset \partial X$, *the shadow of A seen from x* to consist of end-points of geodesic rays (if $x \in X$) or lines (if $x \in \partial X$) starting from x and meeting A. We also use the notation $\times_\Delta$ to denote the direct product without the diagonal.

From Mohsen's shadow lemma [35], we conclude that there is R' with

$$\mu_x \times \mu_{gx}(\mathcal{O}_x B(gx, R') \times \mathcal{O}_{gx} B(x, R')) > 0.$$

Since μ_x is supported on $L(\Gamma_0)$ we conclude that there is

$$(z_1, z_2) \in \left(\mathcal{O}_x B(gx, R') \times_\Delta \mathcal{O}_{gx} B(x, R')\right) \cap \left(L(\Gamma_0) \times_\Delta L(\Gamma_0)\right).$$

Moreover, we may assume that (z_1, z_2) has been chosen such that $[z_1, z_2]$ does not pass through the boundary of any rectangle. Too see this, we use that the set of $(y_1, y_2) \in \partial X \times_\Delta \partial X$ such that $[y_1, y_2]$ passes through the boundary of any rectangle gS^i has zero $\mu_x \times \mu_{gx}$-measure (a proof is given in the Axiom A diffeomorphism case [4]). Alternatively, we can use the property that the rectangles are the closure of their interior.

By the CAT$(-\kappa)$ property of X, there are $T_1, T_2 > 0$ (depending on R') such that $[z_1, z_2]$ passes within T_1 of x and gx, provided $d(x, gx) \geq T_1$. For a proof of this statement, see Lemma 3.17 of [35]. For those finitely many $g \in \Gamma_0$ with $d(x, gx) < T_1$, we choose $D > 0$ such that $D \geq d(gx, [z_1, z_2]) + d(x, [z_1, z_2])$ (recall that z_1, z_2 are functions of g). Thus taking $R = \max(D, T_2)$ completes the proof of the claim. $\square$

It was stated in section 3.3 that weak symmetry for the one-sided shift space is equivalent to a condition involving only periodic points. We include a proof here.

Lemma 3.6.2. *A Hölder continuous $f : \Sigma^+ \to \mathbb{R}$ is weakly symmetric with respect to † if and only if there are $\beta(n)$ with $\beta(n)/n \to 0$ as $n \to \infty$ such that $|r^n(x) - r^n(x^\dagger)| \leq \beta(n)$ for any $x \in \Sigma^+$ with $\sigma^n x = x$, and $x^\dagger \in \Sigma^+$ defined by $(x^\dagger)^i = (x^{kn-i})^\dagger$, for $i = 0, \ldots, n-1$ and $k \in \mathbb{N}$.*

Proof. Suppose that $f : \Sigma^+ \to \mathbb{R}$ is weakly symmetric with respect to †. Let $x \in \Sigma^+$ with $\sigma^n x = x$, and write $w = x^0 \ldots x^{n-1} \in \mathcal{W}^n$. Then $x \in [w]$ and $x^\dagger \in [w^\dagger]$ and so $|r^n(x) - r^n(x^\dagger)| \leq \log D_n$ by hypothesis.

For the converse, let $w \in \mathcal{W}^n$. By the aperiodicity of Σ^+, there is $u \in \mathcal{W}^p$ (where p is the aperiodicity constant) such that wuw is admissible. Define $x \in \Sigma^+$ by the infinite concatenation of wu. Then for any $y \in [w]$, and

$z \in [w^\dagger]$, we have $|r^n(y) - r^n(x)|, |r^n(z) - r^n(\sigma^p x^\dagger)| \le |r|_\theta/(1-\theta)$. Therefore,

$$\begin{aligned}
&|r^n(y) - r^n(z)| \\
&\le |r^n(z) - r^n(\sigma^p x^\dagger)| + |r^n(y) - r^n(x)| \\
&+ |r^{n+p}(x) - r^{n+p}(x^\dagger)||r^p(\sigma^n x) - r^p(x^\dagger)| \\
&\le C + \beta(n+p).
\end{aligned}$$

For some constant $C > 0$ independent of w and n. The result follows by setting $D_n = e^C e^{\beta(n+p)}$. □

It is notable that the function given in Proposition 3.5.2 and Lemma 3.6.1 concern the two-sided shift space, whereas Theorem 3.4.1 is for the one-sided shift space. We can relate these in the following lemma. We say that a function, $f : \Sigma \to \mathbb{R}$, *depends only on future coordinates* if $f(x) = f(y)$ when $x^i = y^i$ for all $i \in \mathbb{Z}_{\ge 0}$. In this way, we may consider f to be a function $f : \Sigma^+ \to \mathbb{R}$.

Lemma 3.6.3. *For any Hölder continuous $r : \Sigma \to \mathbb{R}$, there is a Hölder continuous $r' : \Sigma \to \mathbb{R}$ depending only on future coordinates, satisfying*

$$\sum_{i=0}^{n-1} r'(\sigma^i x) = \sum_{i=0}^{n-1} r(\sigma^i x)$$

for any $x \in \Sigma$ with $\sigma^n x = x$. Moreover, if (r, ψ) is weakly symmetric, then (r', ψ) is weakly symmetric, and we have $P_{\mathrm{Gur}}(-sr', T_{\psi_\Gamma}) = P_{\mathrm{Gur}}(-sr, T_{\psi_\Gamma})$.

Proof. First, we can find a Hölder continuous $r' : \Sigma \to \mathbb{R}$ depending on future coordinates which is cohomologous to r. Then, we note that since r and r' are cohomologous, it follows that

$$\sum_{i=0}^{n-1} r'(\sigma^i x) = \sum_{i=0}^{n-1} r(\sigma^i x)$$

for any $x \in \Sigma$ with $\sigma^n x = x$. A good reference for this material is [33].

The second statement follows easily from the first, as only periodic points appear in definition of the Gurevič pressure, and periodic points are sufficient to verify weak symmetry by Lemma 3.6.2. □

Remark 3.6.1. Note that in the lemma above, r' may have a different Hölder exponent to r.

We are now ready to prove the main theorem.

Proof of Theorem 3.1.1. We begin with a proof of the following:

$$\inf_{\Gamma\in\mathcal{N}} \kappa_A(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0 \implies \sup_{\Gamma\in\mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0}.$$

Let $\mathcal{N}$ be an arbitrary collection of normal subgroups of Γ_0 with

$$\inf_{\Gamma\in\mathcal{N}} \kappa_A(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0.$$

Write $\kappa = \inf_{\Gamma\in\mathcal{N}} \kappa_A(\pi_{\Gamma_0/\Gamma}, \mathbb{1})$. Since the trivial group $\{1\}$ has $\delta_{\Gamma_0} > \delta_{\{1\}}$ by construction, we will assume from now that $\{1\} \notin \mathcal{N}$.

By Lemma 3.6.1, in order to get a uniform bound on δ_Γ, we just need to give a uniform bound for $s = h_\Gamma$ such that $P_{\mathrm{Gur}}(-sr, T_{\psi_\Gamma}) = 0$. Note that the unique value $s = h_0$ for which $P(-sr, \sigma) = 0$ satisfies $h_0 = h_{\Gamma_0} = \delta_{\Gamma_0}$

Though ψ depends on two letters as opposed to one, we may still apply the conclusion of Theorem 3.4.1(iii). That is, there are $\epsilon_1, \epsilon_2 > 0$ such that for all $s \in [h_0 - \epsilon_1, h_0]$ and all $\Gamma \in \mathcal{N}$,

$$\mathrm{spr}(\mathcal{L}_{-sr',\Gamma}) \leq (1-\epsilon_1)\mathrm{spr}(L_{-h_0 r'}) = (1-\epsilon_2),$$

noting that $\mathrm{spr}(L_{-h_0 r'}) = 1$. Moreover, since T_{ψ_Γ} is transitive when $\Gamma \neq \{1\}$, we have that for all $s \in [h_0 - \epsilon_2, h_0]$,

$$P_{\mathrm{Gur}}(-sr', T_{\psi_\Gamma}) \leq \log \mathrm{spr}(\mathcal{L}_{-sr',\Gamma}).$$

Hence for all $s \in [h_0 - \epsilon_2, h_0]$

$$P_{\mathrm{Gur}}(-sr, T_{\psi_\Gamma}) = P_{\mathrm{Gur}}(-sr', T_{\psi_\Gamma}) \leq \log(1-\epsilon_1) < 0,$$

and so $h_\Gamma \leq h_0 - \epsilon_1$ as required.

We now proceed with the second part, completing the proof of Theorem 3.1.1. That is, we will prove that

$$\inf_{\Gamma\in\mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) = 0 \implies \sup_{\Gamma\in\mathcal{N}} \delta_\Gamma = \delta_{\Gamma_0}.$$

By Theorem 3.4.1(i),

$$\sup_{\Gamma\in\mathcal{N}} P_{\mathrm{Gur}}(-sr', T_{\psi_\Gamma}) = P(-sr', \sigma),$$

for every s. In particular, for $s = h_0$,

$$\sup_{\Gamma \in \mathcal{N}} P_{\mathrm{Gur}}(-h_0 r', T_{\psi_\Gamma}) = 0,$$

and for every $\epsilon > 0$,

$$\sup_{\Gamma \in \mathcal{N}} P_{\mathrm{Gur}}(-(h_0 - \epsilon)r', T_{\psi_\Gamma}) = P(-(h_0 - \epsilon)r, \sigma) < 0.$$

It follows that we can find a sequence Γ_n with

$$P_{\mathrm{Gur}}(-(h_0 - \frac{1}{n})r, T_{\psi_{\Gamma_n}}) < 0.$$

Since $h_0 - 1/n \le h_{\Gamma_n} \le h_0$, we conclude that $h_{\Gamma_n} \to h_0$ as $n \to \infty$; and by Lemma 3.6.1, $\delta_{\Gamma_n} \to \delta_{\Gamma_0}$ as $n \to \infty$. □

3.7 Proof of Theorem 3.4.1(i)

We now return to the setting of subshifts of finite type and their group extensions. Let $\sigma : \Sigma^+ \to \Sigma^+$ be a mixing subshift of finite type and $T_\psi : \Sigma^+ \times G \to \Sigma^+ \times G$. Fix a finite generating set A of G and let $\mathcal{N}$ be a collection of normal subgroups of G.

The aim of this section is to prove the following theorem.

Theorem 3.4.1 (i)**.** *Assume that (ψ, r) is weakly symmetric. Then*

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) = 0 \implies \sup_{H \in \mathcal{N}} P_{\mathrm{Gur}}(r, T_{\psi_H}) = P(r, \sigma).$$

Write ρ_H for the representation of G in $\mathcal{U}(\ell^2(G/H))$ induced by the action of G on the cosets G/H. We have that $\kappa_A(\rho_H, \mathbb{1}) = \kappa_{A/H}(\pi_{G/H}, \mathbb{1})$.

We make use of an argument found in [31] which characterises property (T) in terms of the spectra of G-equivariant symmetric random walks. As we use a particular family of representations, we simplify the result to our setting. Let $\mu : G \to [0, 1]$ be a discrete probability measure with $\mu(g) = \mu(g^{-1})$ for all $g \in G$. In our setting, we always assume that μ has finite support. Define the random walk operator $M : \ell^2(G) \to \ell^2(G)$ by $Mf(x) = \sum_{g \in G} \mu(g) f(xg)$. In this way we can write $M = \sum_{g \in G} \mu(g) \pi_G(g)$. The operator M descends to the quotients of G in a straightforward way: for $H \trianglelefteq G$ define $M_H : \ell^2(G/H) \to \ell^2(G/H)$ by $M_H = \sum_{g \in G} \mu(g) \rho_H(g)$. We write $\mathrm{spr}(M_H)$ for the spectral radius of the operator M_H on the space $\ell^2(G/H)$.

Proposition 3.7.1 (Ollivier [31]). *Let $B = \operatorname{supp}(\mu)$. Then*

$$\sup_{H \in \mathcal{N}} \operatorname{spr}(M_H) < 1 \implies \inf_{H \in \mathcal{N}} \kappa_B(\rho_H, \mathbb{1}) > 0.$$

We present the short proof of this fact.

Proof. Write $\sigma = \sup_{H \in \mathcal{N}} \operatorname{spr}(M_H)$. Suppose that $v \in \ell^2(G/H)$ is ϵ,B-invariant; that is,

$$|\rho_H(b)v - v| \leq \epsilon\, |v|\,,$$

for all $b \in B$. Then $|M_H v - v| \leq \epsilon$. Expanding the norm, and noting the self-adjointness of M_H we have $2\langle v, v\rangle - 2\langle M_H v, v\rangle \leq \epsilon^2$. Rearranging gives that $\langle M_H v, v\rangle \geq \langle v, v\rangle - \epsilon^2/2 = 1 - \epsilon^2/2$. Since $\operatorname{spr}(M_H) = \sup_{f \in \ell^2(G/H), |f|=1} \langle M_H f, f\rangle$ it follows that $\operatorname{spr}(M_H) \geq 1 - \epsilon^2/2$. Therefore $\epsilon^2/2 \geq 1 - \sigma$, and so $\kappa_A(\rho_H, \mathbb{1}) \geq \sqrt{2(1-\sigma)}$. As σ is independent of $H \in \mathcal{N}$, we conclude that $\inf_{H \in \mathcal{N}} \kappa_A(\rho_H, \mathbb{1}) > 0$. □

We are now ready to prove the theorem.

Proof of Theorem 3.4.1(i). Assume that (ψ, r) is weakly symmetric. Assume that

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) = 0.$$

Recall that $\kappa_{A/H}(\pi_{G/H}, \mathbb{1}) = \kappa_A(\rho_H, \mathbb{1})$.

We make use of the following notation. For $a \in \mathcal{W}$, $n \in \mathbb{N}$ and $g \in G$, write

$$\mathcal{W}^n_{a,a^\dagger}(g) = \left\{ u \in \mathcal{W}^n : u^0 = a, ua^\dagger \text{ is admissible}, \psi^n(u) = g \right\},$$

and

$$\mathcal{W}^n_{a,a^\dagger}(g, g^{-1}) = \mathcal{W}^n_{a,a^\dagger}(g) \cup \mathcal{W}^n_{a,a^\dagger}(g^{-1}).$$

Define $\mu_n^{(1)}, \mu_n^{(2)} : G \to [0,1]$ by

$$\mu_n^{(1)}(g) = \frac{\sum_{u_1 \in \mathcal{W}^n_{a,a^\dagger}(g,g^{-1})} e^{r^n(u_1 x_1)}}{2 \sum_{u \in \mathcal{W}^n_{a,a^\dagger}} e^{r^n(u x_1)}},$$

for some $x_1 \in [a^\dagger]$; and

$$\mu_n^{(2)}(g) = \frac{\sum_{u_2 \in \mathcal{W}^n_{a^\dagger,a}(g,g^{-1})} e^{r^n(u_2 x_2)}}{2 \sum_{u \in \mathcal{W}^n_{a^\dagger,a}} e^{r^n(u x_2)}},$$

for some $x_2 \in [a]$.

Define $\mu_{2n} = \mu_n^{(1)} \star \mu_n^{(2)}$, where $\star$ indicates the convolution

$$\mu_{2n}(g) = \sum_{g_1, g_2 \in G: g_1 g_2 = g} \mu_n^{(1)}(g_1)\mu_n^{(2)}(g_2).$$

Define the symmetric random walk operators

$$M_{2n,H} : \ell^2(G/H) \to \ell^2(G/H)$$

$$M_{2n,H} = \sum_{g \in G} \mu_{2n}(g)\rho_H.$$

The spectral radius can be found to be

$$\operatorname{spr}(M_{2n,H}) = \limsup_{k\to\infty} \langle M_{2n,H}^k \mathbb{1}_{e_{G/H}}, \mathbb{1}_{e_{G/H}} \rangle^{1/k},$$

where $\mathbb{1}_{e_{G/H}} \in \ell^2(G/H)$ is the indicator function on the identity of G/H. Therefore,

$$\log \operatorname{spr}(M_{2n,H}) = \limsup_{k\to\infty} \frac{1}{k} \log \sum_{h\in H} (\mu_{2n})^{\star k}(h).$$

We claim that there is a sequence $C_n > 0$ with $\limsup_{n\to\infty} C_n^{1/n} \to 1$ such that

$$\frac{1}{k} \log \sum_{h\in H} \mu_{2n}^{\star k}(h) \le \frac{1}{k} \log A_{n,k} - \log B_n + \log C_n;$$

where

$$A_{n,k} = \sum_{h\in H} \sum_{u \in \mathcal{W}_{a,a}^{2nk}(h)} e^{r^{2nk}(ux_2)},$$

and

$$B_n = \sum_{u\in\mathcal{W}_{a,a^\dagger}^n} e^{r^n(ux_1)} \sum_{u\in\mathcal{W}_{a,a^\dagger}^n} e^{r^n(ux_1)}.$$

Note that

$$\limsup_{k\to\infty} \frac{1}{k} \log A_{n,k} \le 2nP_{\mathrm{Gur}}(r, T_{\psi_H});$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log B_n = 2P(r,\sigma),$$

since $P(r,\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathcal{W}_{u,v}^n} e^{r^n(wx)}$ for any $u, v \in \mathcal{W}$, $x \in [v]$.

Assuming the claim (whose proof we give later), we have that

$$\frac{1}{n}\operatorname{spr}(M_{2n,H}) \le 2P_{\mathrm{Gur}}(r, T_{\psi_H}) - \frac{1}{n} \log B_n + \frac{1}{n} \log C_n.$$

Write B_{2n} for the support of μ_{2n}. Since A is assumed to generate G, and since B_{2n} is finite, we have that

$$\inf_{H\in\mathcal{N}} \kappa_A(\rho_H, \mathbb{1}) = 0 \implies \inf_{H\in\mathcal{N}} \kappa_{B_{2n}}(\rho_H, \mathbb{1}) = 0,$$

for each n. Therefore $\sup_{H\in\mathcal{N}} \operatorname{spr}(M_{2n,H}) = 1$. We can choose a sequence H_n such that

$$0 = \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{spr}(M_{2n,H_n}).$$

We conclude that

$$0 = \limsup_{n\to\infty} \frac{1}{n} \operatorname{spr}(M_{2n,H_n}) \le \limsup_{n\to\infty} \left(2P_{\mathrm{Gur}}(r, T_{\psi_{H_n}}) - 2P(r,\sigma)\right),$$

i.e. $\sup_n P_{\mathrm{Gur}}(r, T_{\psi_{H_n}}) = P(r,\sigma)$, as required.

We now prove the claim. We have that

$$\sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g,g^{-1})} e^{r^n(u_1x_1)} = \sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g)} e^{r^n(u_1x_1)} + e^{r^n(u_1^\dagger x_1)},$$

and by weak symmetry,

$$\sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g,g^{-1})} e^{r^n(u_1x_1)} \le \sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g)} 2D_n e^{r^n(u_1x_1)}.$$

Therefore,

$$\begin{aligned}
&\sum_{\substack{g_1,\ldots,g_{2l}\in G:\\ g_1\cdots g_{2k}=h}} \prod_{i=1}^k \sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g_i,g_i^{-1})} e^{r^n(u_1x_1)} \sum_{u_2\in\mathcal{W}^n_{a^\dagger,a}(g_{i+1},g_{i+1}^{-1})} e^{r^n(u_2x_2)} \\
&\le \sum_{\substack{g_1,\ldots,g_{2k}\in G:\\ g_1\cdots g_{2k}=h}} \prod_{i=1}^k \sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g_i)} 2D_n e^{r^n(u_1x_1)} \sum_{u_2\in\mathcal{W}^n_{a^\dagger,a}(g_{i+1})} 2D_n e^{r^n(u_2x_2)} \\
&= (2D_n)^{2k} \sum_{\substack{g_1,\ldots,g_{2l}\in G:\\ g_1\cdots g_{2k}=h}} \prod_{i=1}^k \sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g_i)} e^{r^n(u_1x_1)} \sum_{u_2\in\mathcal{W}^n_{a^\dagger,a}(g_{i+1})} e^{r^n(u_2x_2)}.
\end{aligned}$$

Using the Hölder property of r, for each i we have,

$$\begin{aligned}
&\sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g_i)}\sum_{u_2\in\mathcal{W}^n_{a^\dagger,a}(g_{i+1})} e^{r^n(u_1x_1)}e^{r^n(u_2x_2)}\\
&\le \exp\left(|r|_\theta\sum_{j=0}^n\theta^j\right)\sum_{u\in\mathcal{W}^n_{a,a}(g_ig_{i+1})} e^{r^{2n}(ux_2)}\\
&\le c_\theta\sum_{u\in\mathcal{W}^n_{a,a}(g_ig_{i+1})} e^{r^{2n}(ux_2)},
\end{aligned}$$

where $c_\theta = \exp\left(|r|_\theta\sum_{j=0}^\infty\theta^j\right)$. We then bound the k-length product as

$$\prod_{i=1}^k c_\theta\sum_{u\in\mathcal{W}^n_{a,a}(g_ig_{i+1})} e^{r^{2n}(ux_2)} \le c_\theta^{2k}\sum_{u\in\mathcal{W}^n_{a,a}(h)} e^{r^{2n}(ux_2)}.$$

Putting these bounds together gives

$$\begin{aligned}
&\frac{1}{k}\log\sum_{h\in H}\mu_{2n}^{\star k}(h)\\
&=\frac{1}{k}\log\left(\sum_{h\in H}\sum_{\substack{g_1,\dots,g_{2k}\in G:\\ g_1\cdots g_{2k}=h}}\prod_{i=1}^k\sum_{u_1\in\mathcal{W}^n_{a,a^\dagger}(g_i,g_i^{-1})} e^{r^n(u_1x_1)}\sum_{u_2\in\mathcal{W}^n_{a^\dagger,a}(g_{i+1}g_{i+1}^{-1})} e^{r^n(u_2x_2)}\right)\\
&-\log\sum_{u\in\mathcal{W}^n_{a,a^\dagger}} e^{r^n(ux_1)}\sum_{u\in\mathcal{W}^n_{a,a^\dagger}} e^{r^n(ux_1)}\\
&\le\frac{1}{k}\log\sum_{h\in H}\sum_{u\in\mathcal{W}^{2nk}_{a,a}(h)} e^{r^{2nk}(ux_2)}-\log B_n+2\log 2D_n+2\log c_\theta.
\end{aligned}$$

Writing $C_n = (2D_nc_\theta)^2$, concludes the proof of the claim. $\square$

3.8 Auxiliary lemmas

Recall that L_r has an isolated simple maximum eigenvalue at $e^{P(r,\sigma)}$ and strictly positive eigenfunction $h\in F_\theta$. We say that a function r is *normalised* if $L_r1=1$. Setting $\hat{r}=r-\log h+\log h\circ\sigma-P(r,\sigma)$, we have that $L_{\hat{r}}1=1$, and moreover the spectra are related by $\mathrm{spr}(\mathcal{L}_{r,H}) = e^{P(r,\sigma)}\mathrm{spr}(\mathcal{L}_{\hat{r},H})$. To see this, observe that the following inequality is satisfied pointwise for every H,

$$e^{nP(r)}\frac{\inf_x h(x)}{\sup_x h(x)}(\mathcal{L}_{\hat{r},H})^n \le (\mathcal{L}_{r,H})^n \le e^{nP(r)}\frac{\sup_x h(x)}{\inf_x h(x)}(\mathcal{L}_{\hat{r},H})^n.$$

Therefore it suffices to prove Theorem 3.4.1(ii) under the assumption that r is normalised.

We write $\mathcal{C}^c_H \subset \mathcal{C}^\infty_H$ for the cone of non-negative functions that are constant in the Σ^+-coordinate; that is, $f \in \mathcal{C}^c_H$ if $f(x,g) = f(z,g) \geq 0$ for any $x, z \in \Sigma^+$ and $g \in G/H$. Note that $\mathcal{L}_{r,H}$ does not preserve this cone.

As in the previous section, we write $\rho_H : G \to \mathcal{U}(\ell^2(G/H))$ for the permutation representation determined by $H \trianglelefteq G$. Write $\ell^2_+(G/H)$ for the cone of non-negative functions in $\ell^2(G/H)$.

Lemma 3.8.1. *There exists a constant C such that, for any $H \trianglelefteq G$,*

$$\|(\mathcal{L}_{r,H})^n\| = \sup\{\|(\mathcal{L}_{r,H})^n f\| : f \in \mathcal{C}^c_H, \|f\| = 1\}$$
$$\leq C \sup \left\{ \left\| \sum_{\sigma^n y = z} e^{r^n(y)} \rho_H(\psi^n(y)) f \right\|_{\ell^2(G/H)} : f \in \ell^2_+(G/H), \|f\| = 1, z \in \Sigma^+ \right\}$$

Proof. The first equality is straightforward: for any $f \in \mathcal{C}^\infty_H$, define $\hat{f}(x,g) = \sup_{x \in \Sigma} |f(x,g)|$. Then $\hat{f} \in \mathcal{C}^c_H$ with $\|\hat{f}\| = \|f\|$; and we have $\|(\mathcal{L}_{r,H})^n \hat{f}\| \geq \|(\mathcal{L}_{r,H})^n f\|$.

We now show the second inequality. We have

$$\begin{aligned}\|(\mathcal{L}_{r,H})^n f\| &= \| \sum_{u \in \mathcal{W}} \mathbb{1}_{[u]} (\mathcal{L}_{r,H})^n f\| \\ &\leq \#\mathcal{W} \max_{u \in \mathcal{W}} \sqrt{\sum_{g \in G} \sup_{x \in [u]} |(\mathcal{L}_{r,H})^n f(x,g)|^2}.\end{aligned}$$

Write $u = v$ for the letter attaining this maximum and fix $z \in [v]$. For any $x \in [v]$ and any $f \in \mathcal{C}^c_H$ we have

$$(\mathcal{L}_{r,H})^n f(x,g) = \sum_{\sigma^n y = x} e^{r^n(y)} f(z, \psi^n_H(y) g).$$

In addition,

$$\sup_{x \in [v]} |(\mathcal{L}_{r,H})^n f(x,g)| \leq \exp\left(\frac{|r|_\theta}{1-\theta}\right) \sum_{\sigma^n y = z} e^{r^n(y)} f(z, \psi^n_H(y) g).$$

Writing $\hat{f}(g) = f(z,g)$, we have that $\hat{f}(g) \in \ell^2_+(G/H)$, and

$$(\mathcal{L}_{r,H})^n f(x,g) = \sum_{\sigma^n y = x} e^{r^n(y)} \rho_H(\psi^n(y)) \hat{f}(g).$$

Therefore we have,

$$
\begin{aligned}
\|(\mathcal{L}_{r,H})^n f\| = &\leq \#\mathcal{W} \exp\left(\frac{|r|_\theta}{1-\theta}\right) \sqrt{\sum_{g\in G}\left(\sum_{\sigma^n y = z} e^{r^n(y)} \rho_H(\psi^n(y)) \hat{f}(g)\right)^2} \\
&= C\left|\sum_{\sigma^n y = z} e^{r^n(y)} \rho_H(\psi^n(y)) \hat{f}\right|
\end{aligned}
$$

with $C = \#\mathcal{W} \exp\left(\frac{|r|_\theta}{1-\theta}\right)$. This completes the proof. $\square$

As we have related the spectrum of the group extended transfer operators to expressions involving representations of G in $\ell^2(G/H)$, we will make use of some general results about these representations.

Lemma 3.8.2 (Følner [16]). *If G is non-amenable then:*

$$\forall \epsilon > 0 : \exists B \subset G, B \text{ finite } : \forall E \subset G, E \text{ finite } : \exists b \in B : \#E \cap E \cdot b \leq \epsilon \#E.$$

Proof. In [16] it is shown that G is amenable if and only if there exists ϵ_0 such that for any finite collection $a_1, \ldots, a_n \in G$ there exists a finite $E \subset G$ with

$$\frac{1}{n}\sum_{i=1}^{n} \#(E \cap Ea_i) \geq \epsilon_0 \#E.$$

Negating these statements gives that G is non-amenable if and only if for all ϵ there exists a finite collection $a_1, \ldots, a_n \in G$ such that for every finite $E \subset G$ we have

$$\frac{1}{n}\sum_{i=1}^{n} \#(E \cap Ea_i) \leq \epsilon \#E.$$

And since

$$\frac{1}{n}\sum_{i=1}^{n} \#(E \cap Ea_i) \leq \epsilon \#E \implies \#(E \cap Ea_i) \leq \epsilon \#E \text{ for at least one } a_i$$

the lemma follows. $\square$

Let $\kappa > 0$ and $\rho : G \to \mathcal{U}(\mathcal{H})$ be such that $\kappa_A(\rho, \mathbb{1}) \geq \kappa/2$. Let $f \in \mathcal{H}$ with $|f| = 1$. By the parallelogram law, there is $a \in A$ with

$$|\rho(a)f + f| = 2\sqrt{1 - |\rho(a)f - f|^2/4} \leq 2\sqrt{1 - \kappa^2/16}.$$

We write $\kappa_1 = \sqrt{1 - \kappa^2/16}$.

Lemma 3.8.3. *Let $\epsilon > 0$ be arbitrary. There is a finite subset $B = B(\epsilon) \subset G$, such that the following holds. Assume that $\rho : G \to \mathcal{U}(\mathcal{H})$ with $\kappa_A(\rho, \mathbb{1}) = \kappa > 0$, for some finite subset A of G. For any $f \in (\mathcal{H}, |\cdot|)$ and for each n we may choose $E_n = E_n(f, \epsilon) \subset G$ with the following properties:*

1. $\left|\sum_{g\in E_n} \rho(g)f\right| \leq 2^n(1-\kappa_1)^n |f|$;
2. $\#E_n \geq 2^n(1-\epsilon)^n$
3. *For any $g \in E_n$, we have $g \in (A \cup B)^n$.*

Remark 3.8.1. Notice that though E_n depends on f and on $\mathcal{H}$, the subset B does not depend on f or $\mathcal{H}$.

Proof. Let $\epsilon > 0$ be arbitrary. Let $B = B(\epsilon)$ be the finite set given in Lemma 3.8.3. We may assume that $e \in B$. Let $f \in \mathcal{H}$ be arbitrary. We proceed by induction.

Base case $n = 1$. We may choose $a \in A$ such that

$$|\rho(a)f + f| \geq \kappa |f| \geq \frac{\kappa}{2} |f|,$$

and so

$$|\rho(a)f + f| \leq 2(1-\kappa_1) |f|,$$

satisfying condition 1. Then setting $E_1 = \{a, e\}$ completes the base case of the induction.

Inductive step. Assume the claim is true for n. Set $f_n = \sum_{g\in E_n} \rho(g)f$. Let $a \in A$ such that $|\rho(a)f_n - f_n| \geq \kappa |f_n|$. Let $b \in B$ such that

$$\#(E_n \cup E_n a) \cap (E_n \cup E_n a) \cdot b \leq \epsilon \# E_n a = \epsilon \# E_n.$$

Notice that it follows that

$$\#E_n a \cap E_n ab \leq \epsilon \#(E_n \cup E_n a) \leq 2\epsilon \# E_n,$$

and similarly

$$\#E_n \cap E_n ab \leq 2\epsilon \# E_n.$$

We have two cases to consider:
Case 1. $|\rho(b)\rho(a)f_n - \rho(a)f_n| \geq \frac{\kappa}{2} |f_n|$. The result follows easily setting $E_{n+1} = E_n a \cup E_n ab$.

Case 2. Otherwise,

$$\begin{aligned}
|\rho(ba)f_n - f_n| &= |\rho(ba)f_n - \rho(a)f_n + \rho(a)f_n - f_n| \\
&\geq |\rho(a)f_n - f_n| - |\rho(b)\rho(a)f_n - \rho(a)f_n| \\
&\geq \frac{\kappa}{2}\,|f_n|
\end{aligned}$$

The result follows by setting $E_{n+1} = E_n \cup E_n ab$. $\square$

3.9 Proof of Theorem 3.4.1(ii)

We are now almost in a position to prove the theorem. Therefore, assume that (Σ^+, ψ, G) satisfies (LVR), and write the associated map χ, linear constant L and remainder set $\mathcal{R} \subset G$. Let $\mathcal{N}$ be a collection of normal subgroups of G for which

$$\inf_{H \in \mathcal{N}} \kappa_{A/H}(\pi_{G/H}, \mathbb{1}) = \kappa > 0,$$

for a finite generating set $A \subset G$. Recall from section 3.8 that it suffices to prove the theorem under the condition that r is normalised.

The aim of this section is to find $N_1, N_2 \in \mathbb{N}$ and $\eta(\kappa) > 0$ such that, for any $H \in \mathcal{N}$, for any $f \in \ell^2_+(G/H)$, $|f| = 1$, and any $x \in \Sigma^+$,

$$\left| \sum_{\sigma^{nN_1} y = x} e^{r^{nN_1}(y)} \rho_H(\psi^{nN_1}(y)) f \right| \leq (1 - \eta(\kappa))^{nN_2}.$$

In this case, by the inequality given in Lemma 3.8.1,

$$\begin{aligned}
&\|(\mathcal{L}_{H,r}^{N_1})^n\| \\
&\leq C \sup \left\{ \left| \sum_{\sigma^{nN_1} y = x} e^{r^{nN_1}(y)} \rho_H(\psi^{nN_1}(y)) f \right| : f \in \ell^2_+(G/H), |f| = 1, x \in \Sigma^+ \right\} \\
&\leq C(1 - \eta(\kappa))^{nN_2},
\end{aligned}$$

and so $\mathrm{spr}(\mathcal{L}_{r,H}) \leq (1 - \eta(\kappa))^{\frac{N_2}{N_1}} < 1$, as required.

We proceed with this aim. We simplify the notation by identifying ψ_H with $\rho_H \circ \psi$. In this way our aim is to find

$$\left| \sum_{\sigma^{nN_1} y = x} e^{r^{nN_1}(y)} \psi_H^{nN_1}(y) f \right| \leq (1 - \eta(\kappa))^{nN_2}.$$

Recall that $\kappa_A(\rho_H, \mathbb{1}) = \kappa_{A/H}(\pi_{G/H}, \mathbb{1})$ and so, by hypothesis,

$$\inf_{H\in\mathcal{N}} \kappa_A(\rho_H, \mathbb{1}) = \kappa.$$

Once and for all, fix $\epsilon > 0$ sufficiently small to satisfy the following inequality

$$(1-\epsilon) > (1-\kappa_1),$$

where $\kappa_1 = \sqrt{1-\kappa^2/16}$, as in the previous section. Now that ϵ is fixed, the set $B = B(\epsilon)$ from Lemma 3.8.3 is fixed.

The following constants appear in the estimations: $\alpha = \min_{x\in\Sigma^+} e^{r(x)}$, p is the aperiodicity constant for Σ^+, $W = \#\mathcal{W}, R = \#\mathcal{R}$, L is the (LVR) linear constant, and we write $K = \max_{g\in A\cup B\cup\mathcal{R}} k_g$, where k_g denotes the length of the word $\chi(g)$. With these definitions we have $\psi^{k_g}(\chi(g)) = r_0(g)gr_1(g)$ with $k_g \leq mKL$, for some $r_0(g), r_1(g) \in \mathcal{R}$, for each m, and each $g \in (A\cup B)^m$.

Fix m sufficiently large to satisfy the inequality

$$\frac{(1-\epsilon)^m}{(1-\kappa_1)^m} > \exp\frac{|r|_\theta}{1-\theta}(mKLRW)^2.$$

It will be useful to write

$$\kappa_2 = \frac{1}{mKL}(1-\epsilon)^m - \exp\frac{|r|_\theta}{1-\theta}(RW)^2 mKL(1-\kappa_1)^m > 0.$$

Lemma 3.9.1. *Let $H \in \mathcal{N}$ be arbitrary. For every $f \in \ell^2_+(G/H)$, $|f| = 1$, and every $v_0, v_1 \in \mathcal{W}$ there exists $P_{v_0,v_1} \subset \mathcal{W}^{mKL+2p}_{v_0}$ such that*

$$\left|\sum_{w\in P_{v_0,v_1}} \psi_H^{mKL+2p}(w) f\right| \leq 2^m((RW)^2 mKL(1-\kappa_1)^m),$$

and

$$\#P_{v_0,v_1} \geq \frac{1}{mKL}2^m(1-\epsilon)^m.$$

Proof. We describe a procedure to sandwich an arbitrary word w between any two letters v_0, v_1; and then apply this to $w_g = \chi(g)$.

Let $v_0, v_1 \in \mathcal{W}$ be arbitrary. For each $v \in \mathcal{W}$ fix $u_0(v) \in \mathcal{W}^p$ with initial letter $(u_0(v))^0 = v_0$, and such that $u_0(v)v$ is admissible. For each $0 \leq s \leq mKL$, and each $v \in \mathcal{W}$, fix $u_1(v,s) \in \mathcal{W}^{mKL+p-s}$ with initial letter $(u_1(v,s))^0 = v$ and such that $u_1(v,s)v_1$ is admissible.

Let $H \in \mathcal{N}$ be arbitrary, and let $f \in \ell^2_+(G/H)$. Fix

$$E_m = E_m\left(\sum_{r\in\mathcal{R}} \rho_H(r) \sum_{\substack{j\in\mathcal{W},\\ 0<s\le mKL+p}} \psi_H^{mKL+p-s}(u_1(s,j))f\right).$$

Define P_{v_0,v_1} to be the collection of words $p_g = u_1(w_g^0, k_g)w_g u_0(w_g^{k_g-1})$, $g \in E_m$ (recall that w_g^0, $w_g^{k_g-1}$ denote the initial and terminal letters of w_g respectively). Then $P_{v_0,v_1} \subset \mathcal{W}^{mKL+2p}$.

The map $g \mapsto p_g$ is not necessarily injective, as if w_{g_1} is nested in w_{g_2} then it is possible to choose $u_1(w_g^0, k_g)$, for $g = g_1, g_2$, such that $p_{g_1} = p_{g_2}$. However, this is the only way in which the map fails to be injective: if w_{g_1} (respectively, w_{g_2}) is not nested in w_{g_2} (respectively, w_{g_1}), then $p_{g_1} \neq p_{g_2}$. Since the length of $u_1(w_g^0, k_g)$ is at most mKL, it follows that, for a fixed g_1, $p_{g_1} = p_{g_2}$ for at most mKL distinct g_2. Thus,

$$\#P_{v_0,v_1} \ge \frac{1}{mKL}\#\{w_g : g \in E_m\} \ge \frac{1}{mKL}2^m(1-\epsilon)^m.$$

Since $f \ge 0$, we can make the following estimations,

$$\begin{aligned}
&\left|\sum_{w\in P_{v_0,v_1}} \psi_H^{mKL+2p}(w)f\right| \\
&= \left|\sum_{g\in E_m} \psi_H^p(u_0(w_g^0))\psi_H^{k_g}(w_g)\psi_H^{mKL+p-k_g}(u_1(w_g^{k_g-1},k_g))f\right| \\
&\le \left|\sum_{i\in\mathcal{W}} \psi_H^p(u_0(i)) \sum_{g\in E_m} \rho_H(r_0(g))\rho_H(g)\rho_H(r_1(g)) \sum_{\substack{j\in\mathcal{W},\\ 0<s\le mKL}} \psi_H^{mKL+p-s}(u_1(s,j))f\right| \\
&\le W\left|\sum_{r_0\in\mathcal{R}} \rho_H(r_0) \sum_{g\in E_m} \rho_H(g)\left(\sum_{r_1\in\mathcal{R}} \rho_H(r_1) \sum_{\substack{j\in\mathcal{W},\\ 0<s\le mKL}} \psi_H^{mNq+p-s}(u_1(s,j))f\right)\right| \\
&\le RW2^m(1-\kappa_1)^m\left|\sum_{r_1\in\mathcal{R}} \rho_H(r_1) \sum_{\substack{j\in\mathcal{W},\\ 0<s\le mKL}} \psi_H^{mKL+p-s}(u_1(s,j))f\right| \\
&\le (RW)^2 mKL2^m(1-\kappa_1)^m\,|f|.
\end{aligned}$$

$\square$

For every $x \in [v_0]$ we may extend P_{v_0,v_1} to $P_{x,v_1} \subseteq \sigma^{-(mKL+2p)}x$. We informally refer to elements of P_{x,v_1} as *paths* (from v_1 to x). We are now ready to prove the theorem.

Proof of Theorem 3.4.1(ii). For simplicity, we write $M = mKL + 2p$. In order to prove the theorem recall it suffices to show that,

$$\left| \sum_{\sigma^{nM}y=x} e^{r^{nM}(y)} \psi_H^{nM}(y) f \right| \leq (1 - \alpha^M 2^m \kappa_2)^{nm}$$

for any $H \in \mathcal{N}$, any $f \in \ell^2_+(G/H)$ with $|f| = 1$, and for any $x \in \Sigma^+$. Note that α, m, M, κ_2 do not depend on H or on f.

Base case. Let $x \in \Sigma^+$ be given. Fix some $v_1 \in \mathcal{W}$ and let P_{x,v_1} be given by Lemma 3.9.1. We have,

$$\begin{aligned}
&\left| \sum_{\sigma^{nM}y=x} e^{r^{nM}(y)} \psi_H^{nM}(y) f \right| \\
&\leq \left| \sum_{w \in \mathcal{W}_x^M} (e^{r^M(wx)} - \alpha^M \mathbb{1}_{P_{x,v_1}}(w)) \psi_H^M(w) f \right| + \left| \sum_{w \in \mathcal{W}_x^M, w \in P_{x,v_1}} \alpha^M \psi_H^M(w) f \right| \\
&\leq \left(1 - \alpha^M \# P_{x,v_1}\right) + \alpha^M \left| \sum_{w \in \mathcal{W}_x^M, w \in P_{x,v_1}} \psi_H^M(w) f \right| \\
&\leq \left(1 - \frac{1}{mKL} 2^m (1-\epsilon)^m \alpha^M\right) + \alpha^M (RW)^2 mKL 2^m (1-\kappa_1)^m \\
&= 1 - \alpha^M 2^m \left(\frac{1}{mKL}(1-\epsilon)^m - (RW)^2 mKL(1-\kappa_1)^m\right) \\
&\leq 1 - \alpha^M 2^m \kappa_2.
\end{aligned}$$

Inductive step. Assume that for every $f \in \ell^2_+(G/H)$, and every $x \in \Sigma^+$,

$$\left| \sum_{\sigma^{nM}y=x} e^{r^{nM}(y)} \psi_H^{nM}(y) f \right| \leq (1 - \alpha^M \kappa_2)^{nm}.$$

We will show that for every $f \in \ell^2_+(G/H)$ and every $x \in \Sigma^+$,

$$\left| \sum_{\sigma^{(n+1)M}y=x} e^{r^{(n+1)M}(y)} \psi_H^{(n+1)M}(y) f \right| \leq (1 - \alpha^M \kappa_2)^{(n+1)m}.$$

With this aim, let $f \in \ell^2_+(G/H)$ and $x \in \Sigma^+$ be arbitrary. Fix some $v_1 \in \mathcal{W}$

and $\hat{x} \in [v_1]$. Let P_{x,v_1} correspond to $E_m = E_m\left(\sum_{\sigma^{nM} y = \hat{x}} \psi_H^{nM}(y) f\right)$.

We have

$$\begin{aligned}
&\left|\sum_{\sigma^{(n+1)M} y = x} e^{r^{(n+1)M}(y)} \psi_H^{(n+1)M}(y) f\right| \\
&= \left|\sum_{\sigma^M x' = x} \sum_{\sigma^{nM} y = x'} e^{r^M(x')} e^{r^{nM}(y)} \psi_H^{mKL+2}(x') \psi_H^{nM}(y) f\right| \\
&\leq \left|\sum_{\sigma^M x' = x} \sum_{\sigma^{nM} y = x'} (e^{r^M(x')} - \alpha^M \mathbb{1}_{P_{x,v_1}}(x')) e^{r^{nM}(y)} \psi_H^M(x') \psi_H^{nM}(y) f\right| \\
&+ \left|\sum_{x' \in P_{x,v_1}} \sum_{\sigma^{nM} y = x'} \alpha^M e^{r^{nM}(y)} \psi_H^M(x') \psi_H^{nM}(y) f\right|.
\end{aligned}$$

We estimate the second term as

$$\begin{aligned}
&\left|\sum_{x' \in P_{x,v_1}} \sum_{\sigma^{nM} y = x'} \alpha^M e^{r^{nM}(y)} \psi_H^M(x') \psi_H^{nM}(y) f\right| \\
&\leq \alpha^M \exp\left(\sum_{i=1}^M |r|_\theta \theta^i\right) \left|\sum_{x' \in P_{x,v_1}} \psi_H^M(x') \sum_{\sigma^{nM} y = \hat{x}} e^{r^{nM}(y)} \psi_H^{nM}(y) f\right| \\
&\leq \alpha^M \exp\left(\frac{|r|_\theta}{1-\theta}\right) (RW)^2 mKL 2^m (1-\kappa_1)^m \left|\sum_{\sigma^{nM} y = \hat{x}} e^{r^{nM}(y)} \psi_H^{nM}(y) f\right|.
\end{aligned}$$

Write $x_{\max}$ for the element in Σ^+ maximizing

$$z \mapsto \left|\sum_{\sigma^{nM} y = z} e^{r^{nM}(z)} \psi_H^{nM}(y) f\right|.$$

We have,

$$
\begin{aligned}
&\left|\sum_{\sigma^M x'=x}\sum_{\sigma^{nM} y=x'}(e^{r^M(x')}-\alpha^M \mathbb{1}_{P_{x,v_1}}(x'))e^{r^{nM}(y)}\psi_H^M(x')\psi^{nM}(y)f\right| \\
&\le \sum_{\sigma^M x'=x}(e^{r^M(x')}-\alpha^M \mathbb{1}_{P_{x,v_1}}(x'))\left|\sum_{\sigma^{nM} y=x'} e^{r^{nM}(y)}\psi_H^{nM}(y)f\right| \\
&\le \sum_{\sigma^M x'=x}(e^{r^M(x')}-\alpha^M \mathbb{1}_{P_{x,v_1}}(x'))\left|\sum_{\sigma^{nM} y=x_{\max}} e^{r^{nM}(y)}\psi_H^{nM}(y)f\right| \\
&\le (1-\alpha^M\frac{1}{mKL}2^m(1-\epsilon)^m)\left|\sum_{\sigma^{nM} y=x_{\max}} e^{r^{nM}(y)}\psi_H^{nM}(y)f\right|.
\end{aligned}
$$

We therefore conclude the inductive step,

$$
\begin{aligned}
&\left|\sum_{\sigma^{(n+1)M} y=x} e^{r^{(n+1)M}(y)}\psi_H^{(n+1)M}(y)f\right| \\
&\le \left(\alpha^M \exp\left(\frac{|r|_\theta}{1-\theta}\right)(RW)^2 mKL2^m(1-\kappa_1)^m+1-\frac{1}{mKL}2^m(1-\epsilon)^m\alpha^M\right) \\
&\times \left|\sum_{\sigma^{nM} y=x_{\max}} e^{r^{nM}(y)}\psi_H^{nM}(y)f\right| \\
&\le (1-\alpha^M 2^m\kappa_2)\left|\sum_{\sigma^{nM} y=x_{\max}} e^{r^{nM}(y)}\psi_H^{nM}(y)f\right|.
\end{aligned}
$$

□

3.10 Proof Theorem 3.4.1(iii)

As in section 3.8, we normalise a Hölder function f by setting $\hat{f} = f - \log h + \log h\circ\sigma - P(f,\sigma)$, where h is the eigenfunction (of maximal eigenvalue) for L_f. Let $s\mapsto r_s \in F_\theta$ be continuous (in the $\|\cdot\|_\theta$ topology) for $s\in[-1,1]$. Write h_s for the maximal eigenfunction for L_{r_s}. Since the maximal eigenvalue is simple and isolated for all s, it follows that $s\mapsto h_s$ is continuous for $s\in[-1,1]$; see for instance [23, Chapter 4]. Write $\hat{r}_s = r_s - \log h_s + \log h_s\circ\sigma - P(r_s,\sigma)$. Then it follows that $s\mapsto \hat{r}_s\in F_\theta$ is also continuous for $s\in[-1,1]$.

Proof of Theorem 3.4.1(iii). Assume that $\inf_{H\in\mathcal{N}}\kappa_{A/H}(\pi_{G/H},\mathbb{1}) = \kappa > 0$. As in the previous section, choose $\epsilon < \kappa_1$ and choose m such that

$$\frac{(1-\epsilon)^m}{(1-\kappa_1)^m} > (mKLRW)^2 \exp\left(\frac{|\hat{r}_0|_\theta}{1-\theta}\right).$$

Since $s \mapsto \hat{r}_s$ is continuous, we may choose $\delta > 0$ such that

$$\frac{(1-\epsilon)^m}{(1-\kappa)^m} > (mKLRW)^2 \exp\left(\frac{|\hat{r}_s|_\theta}{1-\theta}\right),$$

for all $s \in [-\delta, \delta]$. Write

$$\beta = \min_{s\in[-\delta,\delta]} \min_{x\in\Sigma^+} \exp\left(\hat{r}_s(x)\right).$$

and

$$\kappa_2 = \left(\frac{1}{mKL}(1-\epsilon)^m - \left(\max_{s\in[-\delta,\delta]} \exp\frac{|\hat{r}_s|_\theta}{1-\theta}\right)(RW)^2 mKL(1-\kappa_1)^m\right),$$

Note that κ_1 depends only on κ; W and p depend only on Σ^+; and R depends only on ψ. Therefore κ_2 is constant in $H \in \mathcal{N}$ and $s \in [-\delta, \delta]$.

Following the proof of Theorem 3.4.1(ii) we deduce that for each $s \in [-\delta, \delta]$,

$$\mathrm{spr}(\mathcal{L}_{r_s,H}) \leq (1 - \beta^{mKL+2p} 2^m \kappa_2)\mathrm{spr}(L_{r_s}),$$

i.e.

$$\sup_{s\in[-\delta,\delta]} \sup_{H\in\mathcal{N}} \mathrm{spr}(\mathcal{L}_{r_s,H}) < \mathrm{spr}(L_{r_0}).$$

□

Chapter 4

Markov Grammars and critical exponents

We generalise the results of chapters 2 and 3 to CAT(-1) spaces. We use symbolic dynamics given by the Markov grammar for Γ_0, as formulated by Bourdon [3], Lalley [27], and Pollicott and Sharp [39, 40]. We show that, for those Γ_0 with a gregarious Markov grammar, we have $\delta_\Gamma < \delta_{\Gamma_0}$ when Γ_0/Γ is non-amanable; and moreover there is a gap uniform in Γ if and only if the Kazhdan distances of the permutation representation of Γ_0/Γ are bounded away from 0.

4.1 Introduction

Let X be a CAT(-1) space and let Γ_0 be a non-elementary cocompact group of isometries acting freely and properly discontinuously on X. That is, X is a (simply connected) complete geodesic metric space in which every triangle is, roughly speaking, as pinched as the corresponding triangle in hyperbolic space of constant curvature -1. This generalises the setting of chapter 2 and 3: any complete, simply connected Riemannian manifold of curvature bounded above by -1 is CAT(-1). Many constructions in the smooth case pass to the CAT(-1) case; for instance the visual boundary ∂X, limit sets and critical exponents. Since Γ_0 is non-elementary, the cardinality of ∂X is greater than 2 (and indeed is uncountably infinite).

Let Γ be a normal subgroup of Γ_0. For some choice of $x \in X$, the *Poincaré series* $\eta_\Gamma(s)$ is defined to be,

$$\eta_\Gamma(s) = \sum_{g \in \Gamma} e^{-sd(x,gx)},$$

where it converges. In this setting, we will make use of the following characterisation of the *critical exponent* δ_Γ of Γ,

$$\delta_\Gamma = \inf \{ s \in \mathbb{R} : \eta_\Gamma(s) < \infty \}.$$

(The definition is independent of the choice of $x \in X$.) We have the relation,

$$\delta_\Gamma = \limsup_{T\to\infty} \frac{1}{T} \log \#\{g \in \Gamma : d(x, gx) \le T\};$$

and for Γ_0 the limit exists and $\delta_{\Gamma_0} > 0$.

An important case of a CAT(-1) space is the Cayley graph X of the free group on $d \ge 2$ generators (with respect to its symmetric free basis S). Writing $|g|_S$ for the minimal length of the element g expressed as a word in S, we have that $d(g,h) = |g^{-1}h|_S$ defines a CAT(-1) metric on X. In this case, $\delta_{F_d} = 2d - 1$, and for any $H \trianglelefteq F_d$,

$$\delta_H = \limsup_{N\to\infty} \frac{1}{N} \log \#\{g \in H : |g|_S \le N\},$$

is the *relative growth* of H in F_d. Then a theorem of Grigorchuk [18] states that $\delta_H = \delta_{F_d}$ if and only if F_d/H is amenable. Moreover, it was conjectured [19] that the same result should hold for the Cayley graph of any hyperbolic group.

We return to the general setting where X is CAT(-1). We have seen in chapter 2, that when X is a Riemannian manifold, equality of critical exponents $\delta_\Gamma = \delta_{\Gamma_0}$ holds precisely when Γ_0/Γ is amenable. In the CAT(-1) setting, the result of Roblin [43] still applies: if Γ_0/Γ is amenable then $\delta_\Gamma = \delta_{\Gamma_0}$. This conclusion was also obtained by Sharp [48] in the case that Γ_0 is a free group. It is the latter approach that we wish to extend for a particular class of groups, which we describe now.

Since Γ_0 acts co-compactly on a CAT(-1) space, it follows that Γ_0 is word hyperbolic. In this way we may associate to it a Markov grammar, which is roughly speaking, a (finite) directed graph, with a distinguished initial state, which writes the elements of Γ_0 uniquely (and with minimal length). Following Pollicott-Sharp [39], we say that the Markov grammar is *gregarious* if the directed graph has only one connected component that is not a singleton. These definitions will be made precise in section 4.2.

Theorem 4.1.1. *Let Γ_0 be a cocompact group of isometries of a* CAT(-1) *space. Assume that Γ_0 admits a Markov grammar that is gregarious. For any* *collection $\mathcal{N}$ of normal subgroups of Γ_0, we have*

$$\inf_{\Gamma\in\mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0 \implies \sup_{\Gamma\in\mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0}.$$

In particular, for any normal subgroup $\Gamma \leq \Gamma_0$, we have that $\delta_\Gamma = \delta_{\Gamma_0}$ if and only if Γ_0/Γ is amenable

Remark 4.1.1. The class of groups with a Markov grammar that is gregarious includes the class of all cocompact Fuchsian groups [39], [47].

4.2 Background and history

The classical examples of word hyperbolic groups are the cocompact discrete groups of isometries of n-dimensional hyperbolic space. Seminal work of Cannon [9] shows that the growth series $f_S(x) = \sum_{n=0}^{\infty} N(n)x^n$, $N(n) = \#\{g \in \Gamma_0 : |g|_S = n\}$, of such a group G is a rational function. The key tool was to develop a 'linear recursion' for writing the elements G – or in modern terminology, a Markov grammar. Following Bourdon [3] and Lalley [27], this machinery was employed by Pollicott and Sharp [39] to study comparison theorems between the word length and geometric displacement of a point. And later, in the case of variable negative curvature, Pollicott and Sharp [40] showed that the Poincaré series of a discrete cocompact group of isometries Γ_0 has a meromorphic continuation to an $\epsilon > 0$ neighbourhood of the half-plane $\mathrm{Re}(s) > \delta_{\Gamma_0}$. (And this result extends to the CAT(-1) setting.)

We now describe the terminology of Markov grammars. See [17] for further background. In the following, G is an arbitrary finitely generated group, and S a symmetric generating set. Recall that $|g|_S$ denotes the length of the element g expressed in the generators S.

Definition 4.2.1. A *(strong) Markov grammar* (with respect to S) is a (finite) directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with distinguished vertex $\star \in \mathcal{V}$, and function $\lambda : \mathcal{E} \to S$ such that:

1. for every $g \in G$, there is a unique directed path $w_g = e_0, e_1, \ldots, e_{n-1}$ originating from $\star$, satisfying $\lambda(e_{n-1}) \cdots \lambda(e_1)\lambda(e_0) = g$;

2. $w_g = e_0, e_1, \ldots, e_{n-1}$ has its length $n = |g|_S$.

In some literature, the term Markov grammar refers only to the first condition. However, for brevity, we will use it to include the stronger definition given here. We refer to $\star$ as the *initial state*.

Not all groups admit a Markov grammar. The following examples are important to consider in our analysis in section 4.4.

Example 4.1. We construct a Markov grammar for the free group F_2 with generators a, b, a^{-1}, b^{-1} in the following way. The states are

$$\star, v(a), v(b), v(a^{-1}), v(b^{-1}),$$

and allowed transitions

1. $\star, v(g)$ with $\lambda(\star, v(g)) = g$; and
2. $v(h), v(g)$ with $\lambda(v(h), v(g)) = g$, for each $h \neq g^{-1}$, and all $g = a, b, a^{-1}, b^{-1}$.

Example 4.2. If G_1, G_2 are groups with Markov grammars $\mathcal{G}_1, \mathcal{G}_2$, then we can construct a Markov grammar for the direct product $G_1 \times G_2$, simply by treating every vertex in $\mathcal{G}_1$ as the initial state for the Markov grammar $\mathcal{G}_2$. In this way we construct a Markov grammar for $F_2 \times F_2$. Notice that the relative growth of $F_2 \times \{e\}$ is equal to the growth of $F_2 \times F_2$, despite the fact that $(F_2 \times F_2)/(F_2 \times \{e\})$ is non-amenable. This does not contradict our theorem as $F_2 \times F_2$ is not word hyperbolic.

Fortunately, there is a wide class of groups that do admit a Markov grammar.

Proposition 4.2.1. *Word hyperbolic groups admit a Markov grammar for every S.*

In the remainder, $\mathcal{G}$ is the Markov grammar for Γ_0 with respect to some S.

There is a natural equivalence relation $\sim$ on the vertices $\mathcal{V}$ of a directed graph $\mathcal{G}$ given by $i \sim j$ when there is a directed path from i to j and from j to i; or $i = j$. We refer to the $\mathcal{G}$ restricted to each $\sim$ partition element as *a connected component*. In general, a Markov grammar may have multiple connected components (note that the initial state always forms an isolated connected component). Recall that we say that the Markov grammar is *gregarious* if the directed graph has only one connected component that is not a singleton, and that as stated in Remark 4.1.1, this includes all cocompact Fuchsian groups.

We turn the Markov grammar into a subshift of finite type in the following way. Let $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ be the directed graph obtained by adding a 'sink' state 0 to $\mathcal{G}$. That is, $\mathcal{V}_0 = \mathcal{V} \cup \{0\}$, $\mathcal{E}_0 = \mathcal{E} \cup \{(i, 0) : i \in \mathcal{V}_0\}$. Let B be the vertex-transition matrix given by $\mathcal{G}_0$ and $\sigma : \Sigma_B \to \Sigma_B$ the corresponding subshift of finite type with alphabet $\mathcal{V}_0$.

We can relate the geometry of the action of Γ_0 to the statistics of a Hölder continuous function defined on Σ_B^+ in the following way.

Proposition 4.2.2 (Pollicott-Sharp [40])**.** *There is a Hölder continuous $r : \Sigma_B^+ \to \mathbb{R}$ such that*

$$r^n(w_g\dot{0}) = d_X(x, gx),$$

where $w_g = \star, v_1, \ldots, v_n$ is the vertex-path in $\mathcal{G}$ satisfying

$$\lambda(v_n, v_{n-1}) \cdots \lambda(v_2, v_1)\lambda(v_1, \star) = g;$$

and $\dot{0}$ denotes an infinite string of 0s.

Remark 4.2.1. The proposition is originally stated in the case that X is a negatively curved manifold. However, as observed by Sharp [48], the proof only uses properties of the angles of triangles, and so the proposition equally applies in the CAT(-1) setting. The r given by the proposition may not be positive, but is eventually positive, i.e. there is $n > 0$ such that $r^n(x) > 0$ for all $x \in \Sigma_B^+$.

Pollicott and Sharp use Proposition 4.2.2 as a basis for relating spectral properties of the transfer operator to analytic properties of the Poincaré series. We state only a basic result of theirs, which is sufficient for our needs in section 4.4.

4.3 Non-aperiodic subshifts of finite type and group extensions

The previous section gives us an example of a transition matrix that is not irreducible. It is the purpose of this section to recall a well-known decomposition into irreducible components and periodic powers, and show that this passes to group extensions in a similar way.

We update some notation for this section. In the following, Δ is an arbitrary $k \times k$ transition matrix, and $r : \Sigma_\Delta^+ \to \mathbb{R}$ is Hölder continuous. Let $\psi : \Sigma_\Delta^+ \to G$ depend only on the first coordinate. Recall from Chapters 2 and 3 that we define the group extension $T_\psi : \Sigma_\Delta^+ \times G \to \Sigma_\Delta^+ \times G$ by

$$T_\psi(x, g) = (\sigma x, g\psi(x)^{-1}).$$

We use the notation $\{1, \cdots, k\} = \mathcal{W}$; $\mathcal{W}^n(\Delta) \subset \mathcal{W}^n$ for the set of

n-length words w allowed by Δ; $\mathcal{W}_x^n(\Delta) \subset \mathcal{W}^n(\Delta)$ for the subset with wx admissible; and $\mathcal{W}_{a,x}^n(\Delta) \subset \mathcal{W}_x^n(\Delta)$ for the subset with $w^0 = a$.

We write $\mathcal{L}_{r,\Delta}$ for the transfer operator

$$\mathcal{L}_{r,\Delta} f(x,g) = \sum_{w \in \mathcal{W}_x(\Delta)} e^{r(wx)} f(wx, g\psi(w))$$

which is a bounded linear operator on the Banach space

$$\mathcal{C}^\infty(\Sigma_\Delta^+) = \left\{ f \in C(\Sigma_\Delta^+ \times G, \mathbb{R}) : \|f\|_\Delta < \infty \right\},$$

$$\|f\|_\Delta = \sqrt{\sum_{g \in G} \sup_{x \in \Sigma^+} |f(x,g)|^2}.$$

Write $\mathcal{C}^c(\Sigma_\Delta^+)$ for the subspace of functions $f \in \mathcal{C}^\infty(\Sigma_\Delta^+)$ that are constant in the Σ_Δ^+-coordinate; i.e. $f(x,g) = f(y,g)$ for all $x, y \in \Sigma_\Delta^+$ and $g \in G$. In this way, $\mathcal{C}^c(\Sigma_\Delta^+)$ is isomorphic to $\ell^2(G)$.

We say that a $k \times k$ transition matrix Δ' is a submatrix of Δ if $\Delta'(i,j) = 1 \implies \Delta(i,j) = 1$, for any i, j. In this way, we have that $\Sigma_{\Delta'}^+ \subseteq \Sigma_{\Delta'}^+$, and so $r = r_{|\Sigma_{\Delta'}} : \Sigma_{\Delta'}^+ \to \mathbb{R}$ is Hölder continuous.

We will find it useful to use the perspective of directed graphs: we write $\mathcal{G}(\Delta) = (\mathcal{V}, \mathcal{E}(\Delta))$ for the directed graph with vertices $\mathcal{V} = \{1, \ldots, k\}$ and edges $\mathcal{E}(\Delta) = \{(i,j) : \Delta(i,j) = 1\}$. We may partition $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_s$ into connected components. Write $\mathcal{G}_1(\Delta_1), \ldots, \mathcal{G}_s(\Delta_s)$ for the corresponding subgraphs on these vertices, with submatrices $\Delta_1, \ldots, \Delta_s$. We may assume (by relabelling) that $\Delta(u,v) = 1$ for $u \in \mathcal{V}^i, v \in \mathcal{V}^j$ implies that $i \leq j$.

The following extends a standard result for transfer operators of subshifts of finite type (see for instance [39]).

Lemma 4.3.1.

$$\operatorname{spr}(\mathcal{L}_{r,\Delta}, \mathcal{C}^\infty(\Sigma_\Delta^+)) = \max_{i=1,\ldots,s} \operatorname{spr}(\mathcal{L}_{r,\Delta_i}, \mathcal{C}^\infty(\Sigma_{\Delta_i}^+))$$

Proof. We first remark that the inequality

$$\operatorname{spr}(\mathcal{L}_{r,\Delta}, \mathcal{C}^\infty(\Sigma_\Delta^+)) \geq \max_{i=1,\ldots,s} \operatorname{spr}(\mathcal{L}_{r,\Delta_i}, \mathcal{C}^\infty(\Sigma_{\Delta_i}^+)),$$

is easily verified. Therefore we proceed to show the reverse inequality.

For each $N \in \mathbb{N}$, choose ϵ_N such that for each $i = 1, \ldots, s$,

$$\|\mathcal{L}_{r,\Delta_i}^k\|^{1/k} \leq \operatorname{spr}(\mathcal{L}_{r,\Delta_i}) + \epsilon(N)$$

for all $k \geq N$; and conversely, set

$$L_N = \max_{k \leq N} \frac{\|\mathcal{L}^k_{r,\Delta_s}\|}{(\mathrm{spr}(\mathcal{L}_{r,\Delta_s}) + \epsilon_N)^k} \cdots \max_{k \leq N} \frac{\|\mathcal{L}^k_{r,\Delta_1}\|}{(\mathrm{spr}(\mathcal{L}_{r,\Delta_1}) + \epsilon_N)^k}.$$

Since

$$\limsup_{n\to\infty} \|\mathcal{L}^n_{r,\Delta_i}\|^{1/n} = \mathrm{spr}(\mathcal{L}_{r,\Delta_i})$$

we may assume that $\epsilon_N \to 0$ as $N \to \infty$.

For every $w \in \mathcal{W}^n(\Delta)$, we may decompose w uniquely as $w = v_s \cdots v_1$; with $v_i \in \mathcal{W}^{n_i}(\Delta_i)$ for some $0 \leq n_i \leq n$. (Here, $n_i = 0$ corresponds to the empty word.) For each $a \in \mathcal{W}$, fix some $x_a \in \Sigma^+_\Delta$. Define operators $M(\mathcal{W}^k(\Delta_i)) : \ell^2(G) \to \ell^2(G)$ by

$$M(\mathcal{W}^k(\Delta_i)) = \sum_{a \in \mathcal{W}^1(\Delta_i)} \sum_{w \in \mathcal{W}^k_a(\Delta_i)} e^{r^k(wx_a)} \rho \circ \psi^k(w)$$

It is straightforward that $\left|M(\mathcal{W}^k(\Delta_i))\right| \leq \#\mathcal{W}\|\mathcal{L}^k_{r,\Delta_i}\|$.

Let $f \in \mathcal{C}^c_+(\Sigma^+_\Delta) \cong \ell^2_+(G)$ be arbitrary. We have the pointwise inequality

$$\mathcal{L}^n_{r,\Delta} f(x,g) \leq c_L^s \sum_{\substack{n_s,\ldots,n_1: \\ n_s+\cdots+n_1=n}} M(\mathcal{W}^{n_s}(\Delta_s)) \circ \cdots \circ M(\mathcal{W}^{n_1}(\Delta_1)) f(g),$$

where $c_L = \exp(\frac{|r|_\theta}{1-\theta})$. Since all the terms appearing are positive, we conclude that

$$\begin{aligned}
\|\mathcal{L}^n_{r,\Delta} f\| &\leq c_L^s \left| \sum_{\substack{n_s,\ldots,n_1: \\ n_s+\cdots+n_1=n}} M(\mathcal{W}^{n_s}(\Delta_s)) \circ \cdots \circ M(\mathcal{W}^{n_1}(\Delta_1)) f \right| \\
&\leq c_L^s \sum_{\substack{n_s,\ldots,n_1: \\ n_s+\cdots+n_1=n}} |M(\mathcal{W}^{n_s}(\Delta_s)) \circ \cdots \circ M(\mathcal{W}^{n_1}(\Delta_1)) f| \\
&\leq c_L^s \sum_{\substack{n_s,\ldots,n_1: \\ n_s+\cdots+n_1=n}} |M(\mathcal{W}^{n_s}(\Delta_s))| \cdots |M(\mathcal{W}^{n_1}(\Delta_1))| \\
&\leq c_L^s \sum_{\substack{n_s,\ldots,n_1: \\ n_s+\cdots+n_1=n}} \|\mathcal{L}^{n_s}_{r,\Delta_s}\| \cdots \|\mathcal{L}^{n_1}_{r,\Delta_1}\|.
\end{aligned}$$

When $n_i \leq N$ we estimate by L_N, and when $n_i > N$ we estimate by $(\mathrm{spr}\mathcal{L}_{r,\Delta_i} +$

$\epsilon_N)^{n_i}$, to obtain

$$\begin{aligned}
\|\mathcal{L}_{r,\Delta}^n f\| &\le L_N \# \mathcal{W}^s c_L^s \sum_{\substack{n_s,\ldots,n_1: \\ n_s+\cdots+n_1=n}} (\mathrm{spr}\mathcal{L}_{r,\Delta_s} + \epsilon_N)^{n_s} \cdots (\mathrm{spr}\mathcal{L}_{r,\Delta_1} + \epsilon_N)^{n_1} \\
&\le L_N \# \mathcal{W}^s c_L^s n^{s-1} \max_{i=1,\ldots,s} (\mathrm{spr}\mathcal{L}_{r,\Delta_s} + \epsilon_N)^n.
\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \|\mathcal{L}_{r,\Delta}^n\|^{1/n} \le \max_{i=1,\ldots,s} \mathrm{spr}(\mathcal{L}_{r,\Delta_s}) + \epsilon_N.$$

Since $\epsilon_N \to 0$ as $N \to \infty$, we deduce that

$$\mathrm{spr}(\mathcal{L}_{r,\Delta}, \mathcal{C}^\infty(\Sigma_\Delta^+)) \le \max_{i=1,\ldots,s} \mathrm{spr}(\mathcal{L}_{r,\Delta_i}, \mathcal{C}^\infty(\Sigma_{\Delta_i}^+))$$

as required. $\square$

In light of Lemma 4.3.1 it suffices to consider irreducible $k \times k$ matrix Δ in the rest of our analysis. We now show how to decompose Σ_Δ^+ into aperiodic blocks.

For each $i \in \{1, \ldots k\}$, write $d(i)$ for the gcd of all (directed) loops from i to itself. Since Δ is irreducible, this is a constant function $d(i) = d$. Condsider the $kd \times kd$ matrix $\Delta^{(d)}$ defined by $\Delta^{(d)}([v_0, \ldots, v_{d-1}], [v_0', \ldots, v_{d-1}']) = 1$ if and only if $[v_0, \ldots, v_{d-1}], [v_0', \ldots, v_{d-1}']$ are both admissible concatenations by Δ and v_{d-1}, v_0' is admissible by Δ. Then $\Delta^{(d)}$ decomposes into irreducible components $A_1, \cdots, A_d$. We claim that each is aperiodic (indeed, the gcd of loops based at any vertex is equal to 1), and moreover that there are no transitions from a state in A_i to a state in A_j if $i \ne j$. In this way, $\Sigma_{\Delta^{(d)}}^+ = \sqcup_{i=1}^d \Sigma_{A_i}^+$. Moreover, if we consider the inclusions $\Sigma_{A_i}^+ \subseteq \Sigma_\Delta^+$, we may assume that the labelling has been such that for each i $\sigma(\Sigma_{A_i}^+) = \Sigma_{A_{i+1}}^+$ (with the index taken modulo d). We also have that, $\sigma^d : \Sigma_\Delta^+ \to \Sigma_\Delta^+$ is isomorphic to $\sigma : \Sigma_{\Delta^{(d)}}^+ \to \Sigma_{\Delta^{(d)}}^+$. This isomorphism maps $\mathcal{L}_r^d$ to $\mathcal{L}_{r^d}$.

We obtain the following lemma for the group extensions.

Lemma 4.3.2.

$$\mathrm{spr}(\mathcal{L}_{r,\Delta}^d, \mathcal{C}^\infty(\Sigma_\Delta^+)) = \max_{i=1,\ldots,d} \mathrm{spr}(\mathcal{L}_{r^d,A_i}, \mathcal{C}^\infty(\Sigma_{A_i}^+)).$$

Proof. We define a new norm on $\mathcal{C}^\infty(\Sigma_{\Delta^{(d)}}^+)$ by

$$\|f\| = \sum_{i=1}^d \|f \mathbb{1}_{\Sigma_{A_i}^+}\|_{A_i}.$$

Since $\Sigma^+_{\Delta^{(d)}} = \sqcup_{i=1}^d \Sigma^+_{A_i}$ it is clear that $(\mathcal{C}^\infty(\Sigma^+_{\Delta^{(d)}}), \|\cdot\|)$ is isomorphic to the direct sum of Banach spaces $(\mathcal{C}^\infty(\Sigma^+_{A_i}), \|\cdot\|_{\Sigma_{A_i}})$. The transfer operator factors along the direct product as

$$\mathcal{L}^d_{r,\Delta^{(d)}} = \bigoplus_{i=1}^{d} \mathcal{L}_{r^d,A_i}.$$

Observe that $\|\cdot\|$ is an equivalent norm to $\|\cdot\|_{\Delta^{(d)}}$. Since the spectrum of an operator is invariant under equivalent norms it is clear that

$$\mathrm{spr}(\mathcal{L}^d_{r,\Delta^{(d)}}, \mathcal{C}^\infty(\Sigma^+_{\Delta^{(d)}})) = \max_{i=1,\ldots,d} \mathrm{spr}(\mathcal{L}_{r^d,A_i}, \mathcal{C}^\infty(\Sigma^+_{A_i})).$$

□

4.4 Proof of the Theorem

We will make use of the group extension $T_\psi : \Sigma_B \times \Gamma_0 \to \Sigma_B \times \Gamma_0$, with $\psi : \Sigma_B \to \Gamma_0$ defined by $\psi(x) = (\lambda(x^0, x^1))^{-1}$, if $x^1 \neq 0$, and otherwise $\psi(x) = e$. In this way ψ depends only on the first two coordinates. As in Chapter 3, for each $\Gamma \trianglelefteq \Gamma_0$, write ψ_Γ for the projection of ψ to the quotient group Γ_0/Γ, and we write $r : \Sigma_B \times \Gamma_0 \to \mathbb{R}$ for the Γ_0-invariant function which projects to $r : \Sigma^+_B \to \mathbb{R}$ given in proposition 4.2.2.

It will be useful to decorate the transfer operators as $\mathcal{L}_{r,B,\Gamma}$ to indicate both the group Γ, and the matrix B.

We may now relate the Poincaré series to the transfer operators $\mathcal{L}_{r,B,\Gamma}$. We have

$$\begin{aligned}
\eta_\Gamma(s) &= \sum_{g\in\Gamma} e^{-sd(x,gx)} \\
&= \sum_{g\in\Gamma_0} e^{-sd(x,gx)} \mathbb{1}_\Gamma(g) \\
&= \sum_{g\in\Gamma_0} e^{-sr^n(w_g\dot{0})} \mathbb{1}_\Gamma(g) \\
&= \sum_{n\in\mathbb{N}} \sum_{a\in\mathcal{W}, a\neq 0} \sum_{w\in\mathcal{W}^n_{\star,a}} e^{-sr^n(wa\dot{0})} \mathbb{1}_\Gamma(\psi^n(wa)) \\
&= \sum_{a\in\mathcal{W}, a\neq 0,\star} \sum_{n\in\mathbb{N}} \mathcal{L}^n_{-sr,B,\Gamma} \mathbb{1}_{[\star]\times\{e\}}(a\dot{0}, e).
\end{aligned}$$

In particular, for $\Gamma = \Gamma_0$ we see that

$$\eta_{\Gamma_0}(s) = \sum_{a \in \mathcal{W}, a \neq 0, \star} \sum_{n \in \mathbb{N}} L^n_{-sr,B} \mathbb{1}_{[\star]}(a\dot{0}).$$

Let Q be the irreducible component of B corresponding the the connected component of $\mathcal{G}$ that is not a singleton. Write $\Delta_1, \ldots, \Delta_s$ for the remaining irreducible components of B.

Lemma 4.4.1 (Pollicott-Sharp [40])**.** *Let $0 < s \leq \delta_{\Gamma_0}$. We have that*

$$\operatorname{spr}(L_{-sr,Q}, C(\Sigma^+_Q), \|\cdot\|_\infty) \leq 1,$$

with equality $\operatorname{spr}(L_{-sr,Q}, C(\Sigma^+_Q), \|\cdot\|_\infty) = 1$ *precisely when* $s = \delta_{\Gamma_0}$*. For the remaining irreducible components we have*

$$\operatorname{spr}(L_{-sr,\Delta_i}, C(\Sigma^+_{\Delta_i}), \|\cdot\|_\infty) < 1.$$

Proof. Since r is eventually positive, it follows that $-sr^{mn} > 0$ when $s > 0$, for any $m \in \mathbb{N}$. Moreover, since each Δ_i corresponds to a singleton, we have that $\operatorname{spr}(L_{-sr,\Delta_i}, C(\Sigma^+_{\Delta_i}), \|\cdot\|_\infty) < 1$.

For the remainder we will omit reference to the Banach space and simply write $\operatorname{spr}(L_{-sr,B}), \operatorname{spr}(L_{-sr,Q})$ for the spectral radii. Suppose that $0 < s \leq \delta_{\Gamma_0}$. Then by definition of δ_{Γ_0}, we have that $\eta_{\Gamma_0}(s)$ diverges; and so, for some a,

$$1 \leq \limsup_{n\to\infty} (L^n_{-sr,B}\mathbb{1}_{[\star]}(a\dot{0}))^{1/n} \leq \limsup_{n\to\infty} \|L^n_{-sr,B}\mathbb{1}_{[\star]}\|^{1/n}_\infty.$$

Therefore $s < \delta_{\Gamma_0}$ implies that $\operatorname{spr}(L_{-sr,B}) \geq 1$. Since $\operatorname{spr}(L_{-sr,\Delta_i}) < 1$, we conlude by Lemma 4.3.1 that $\operatorname{spr}(L_{-sr,B}) = \operatorname{spr}(L_{-sr,Q}) \geq 1$.

On the other hand, suppose that $s > \delta_{\Gamma_0}$. Then $\eta_{\Gamma_0}(s)$ converges; and so, for all a,

$$\limsup_{n\to\infty} (L^n_{-sr,B}\mathbb{1}_{[\star]}(a\dot{0}))^{1/n} \leq 1.$$

Writing $c_L = \exp(\frac{|r|_\theta}{1-\theta})$, we have that for each $v \in \mathcal{W}(Q)$,

$$L^n_{-sr,B}\mathbb{1}_{[v]}(x_v) \leq c_L L^n_{-sr,B}\mathbb{1}_{[v]}(v\dot{0}),$$

for any $x_v \in [v]$ admissible in Q. And so,

$$\|L^n_{-sr,Q}1\|_\infty \leq c_L \# \mathcal{W} \max_{u \in \mathcal{W}(Q)} L^n_{-sr,Q}\mathbb{1}_{[u]}(u\dot{0}).$$

Let p_v be the length of a path from $\star$ to v. It can be seen that

$$L^n_{-sr,Q}\mathbb{1}_{[v]}(v\dot{0}) \leq \alpha^{-p_v} L^{n+p_v}_{-sr,B}\mathbb{1}_{[\star]}(v\dot{0}),$$

where $\alpha = \min e^{r(x)}$. Therefore, for each $a \neq \star, 0$, we have

$$\limsup_{n\to\infty} \|L^n_{-sr,Q}\|^{1/n} \leq \limsup_{n\to\infty} \left(L^{n+p_a}_{-sr,B}\mathbb{1}_{[\star]}(a\dot{0})\right)^{1/n} \leq 1.$$

Since $s \mapsto \mathrm{spr}(L_{-sr,Q})$ is convex, it follows that $\mathrm{spr}(L_{-sr,Q}) = 1$ precisely when $s = \delta_{\Gamma_0}$. □

We are now ready to prove the theorem.

Theorem 4.4.1. *Let Γ_0 be a cocompact group of isometries of a* CAT(-1) *space. Assume that Γ_0 admits a Markov grammar that is gregarious. For any collection $\mathcal{N}$ of normal subgroups of Γ_0, we have*

$$\inf_{\Gamma\in\mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0 \implies \sup_{\Gamma\in\mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0}.$$

In particular, for any normal subgroup $\Gamma \leq \Gamma_0$, we have that $\delta_\Gamma = \delta_{\Gamma_0}$ if and only if Γ_0/Γ is amenable

Proof. Recall that it is due to Roblin [43] that Γ_0/Γ amenable $\implies \delta_\Gamma = \delta_{\Gamma_0}$. Therefore, in order to prove the theorem, it suffices to show that for any collection $\mathcal{N}$ of normal subgroups of Γ_0, we have

$$\inf_{\Gamma\in\mathcal{N}} \kappa_{A/\Gamma}(\pi_{\Gamma_0/\Gamma}, \mathbb{1}) > 0 \implies \sup_{\Gamma\in\mathcal{N}} \delta_\Gamma < \delta_{\Gamma_0}.$$

(This includes the case where $\mathcal{N} = \{\Gamma\}$, a normal subgroup with non-amenable quotient.)

If Q is not aperiodic, let A be an aperiodic component and let its period be N. It is clear that the induced group extension over Σ_A^+ satisfies (LVR). (To see this, note that $w_g = u_1 u_A u_2$, where u_A is in the aperiodic component, and u_1 and u_2 have length bounded by a constant independent of g.) Therefore we may apply Theorem 3.4.1 to give that there is $0 < \epsilon < \delta_{\Gamma_0}$ such that for all $\Gamma \in \mathcal{N}$ and $s \in (\delta_{\Gamma_0} - \epsilon, \delta_{\Gamma_0} + \epsilon)$,

$$\mathrm{spr}(\mathcal{L}_{-sr^N,A,\Gamma}) < \mathrm{spr}(L_{-\delta_{\Gamma_0} r^N,A}) = 1.$$

Then, by Lemma 4.3.2, and the spectral mapping theorem,

$$\mathrm{spr}(\mathcal{L}_{-sr,Q,\Gamma}) \leq (\mathrm{spr}(\mathcal{L}_{-sr^N,A,\Gamma}))^{1/N} < 1.$$

Recall that, by Lemma 4.3.1,

$$\mathrm{spr}(\mathcal{L}_{-sr,B,\Gamma}) = \max\{\mathrm{spr}(\mathcal{L}_{-sr,Q,\Gamma}), \mathrm{spr}(\mathcal{L}_{-sr,\Delta_i,\Gamma}) : i = 1, \ldots, s\}.$$

For $s \in (\delta_{\Gamma_0} - \epsilon, \delta_{\Gamma_0} + \epsilon)$ we therefore have that

$$\mathrm{spr}(\mathcal{L}_{-sr,B,\Gamma}) < 1,$$

as $\mathrm{spr}(\mathcal{L}_{-sr,\Delta_i,\Gamma}) \le \mathrm{spr}(L_{-sr,\Delta_i}) < 1$.

Therefore, for all $s \in (\delta_{\Gamma_0} - \epsilon, \delta_{\Gamma_0} + \epsilon)$,

$$\eta_\Gamma(s) = \sum_{a \in \mathcal{W}, a \neq 0, \star} \sum_{n \in \mathbb{N}} \mathcal{L}^n_{-sr,B,\Gamma} \mathbb{1}_{[\star]\times\{e\}}(a\dot{0}, e)$$

converges as $\mathrm{spr}(\mathcal{L}_{-sr,B,\Gamma}) < 1$. This implies that $\delta_\Gamma < \delta_{\Gamma_0} - \epsilon$ as required. □

Question 4.3. It was essential to the proof that B have only one connected component that is not a singleton in order to deduce that the restriction to Σ_A^+ satisfies (LVR). Indeed as Example 4.2 shows, this is false for direct products, as each irreducible component may see 'disjoint' parts of the group. However, as remarked, direct products are not word hyperbolic in general, and so one can ask if this problem persists for word hyperbolic groups. More precisely, suppose $Q_1, \ldots, Q_s$ are maximal irreducible components, in the sense that the spectral radius of the transfer operator is attained at each. Can we say anything about the restriction of ψ_Γ to Q_i?

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